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THE EXISTENCE OF CONSERVATION LAWS

BY

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PREFACE

The problem of the existence of conservation laws was suggested to me by Professor Charles Loewner, who has recently made use of some special cases to obtain results of interest in hydrodynamics [3]. The question is also of general interest with regard to the physical interpretation of given systems of partial differential equations. Many important systems arising in physics, Maxwell's equations, for instance, are indeed in the form of conservation laws.

I am indebted to Professor Loewner for his patience, advice, and friendly encouragement throughout the entire writing of this dissertation, to Professor Marcel Riesz, University of Lund, for his interest in the material of Chapter II, and especially to Professor S. S. Chern, University of Chicago, who introduced me to many of the methods used in Chapter III.

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Introduction

A first order quasi-linear homogeneous partial differential equation in the independent variables x^i, $i = 1,\ldots,n$ and the dependent variables u^j, $j = 1,\ldots,p$ is a <u>conservation law</u> if it is of the form

$$(0.1) \qquad \frac{\partial \phi^1}{\partial x^1} + \ldots + \frac{\partial \phi^n}{\partial x^n} = 0$$

for some ϕ^i which are functions of $x = \{x^i\}$ and $u = \{u^j\}$.

If a scalar density $\rho = \rho(x)$ is fixed on the space of x, that is, if a change of variables $x' = x'(x)$ involves $\rho'(x') = \rho(x) \left|\frac{\partial x}{\partial x^1}\right|$, where $\left|\frac{\partial x}{\partial x^1}\right|$ is the Jacobian, then one may set $\phi^i = \rho\tau^i$ and write (0.1) as

$$(0.2) \qquad \frac{1}{\rho}\left(\frac{\partial(\rho\tau^1)}{\partial x^1} + \ldots + \frac{\partial(\rho\tau^n)}{\partial x^n}\right) \equiv \operatorname{div}_\rho \tau = 0,$$

which states that the vector τ with components τ^i is <u>solenoidal with respect to the density</u> ρ. (The notation div_ρ is useful for changes in density; for example, if λ is a scalar, then $\operatorname{div}_{\lambda\rho} \tau = \frac{1}{\lambda} \operatorname{div}_\rho \lambda\tau$.) If a metric $ds^2 = \sum_{ij} g_{ij}(x)dx^i dx^j$ is attached to the space, the vector τ is <u>solenoidal with respect to the given metric</u> when $\rho = \sqrt{|\det g_{ij}|}$ is taken as the scalar density.

In order to investigate whether a given equation has the form (0.2), or in order to investigate whether some linear combination of several equations has this form, which is the purpose of this paper, it is clearly not necessary to know the density ρ . For if one knows only that the equation is in the form of a conservation law (0.1), then for any density ρ the vector $\psi = \frac{1}{\rho}\phi$ is solenoidal with respect to that density. Henceforth a density will be introduced only for purposes of illustration.

Conservation laws arise in several ways in the mechanics of continuous media. If a fluid of density ρ has the velocity vector $u = u(x)$ for x in some closed bounded region R, then Gauss' theorem asserts that

$$(0.3)\qquad \iint_{\partial R} u_n \rho\, d\sigma = \iiint_R \operatorname{div}_\rho u\, \rho\, dx,$$

where u_n is the component of u normal to the boundary ∂R of R. The left-hand side of (0.3) is the variation of the total mass over R. This is physically clear since it gives the normal rate of flow throughout the entire boundary ∂R. If the flow is stationary, then clearly the total mass over any R is conserved, so that the integrand on the right-hand side of (0.3) vanishes, showing that u is solenoidal with respect to ρ .

Variational principles furnish another source of conservation laws. Let f be a function of u^j, u^j_i, and x^i, where $u^j_i = \frac{\partial u^j}{\partial x^i}$, $i = 1,\ldots,n$, $j = 1,\ldots,p$, and suppose that the integral $\int_R f\, dx$, for which a stationary value is sought, is invariant under a group with ρ parameters. Then, according to a well-known result of Emmy Noether, one can find ρ linear combinations of the p resulting variational equations which have the form of conservation laws [4]. As an example, if f does not depend explicitly on the independent variables x^i, then these variables themselves may be taken as parameters. Indeed, by appropriate linear combination of the variational equations

$$(0.4) \qquad f_{u^j} - \sum_{i=1}^{n} \frac{\partial}{\partial x^i} f_{u^j_i} = 0 \qquad (j=1,\ldots,p),$$

one finds

$$(0.5) \qquad \frac{\partial \phi^1_k}{\partial x^1} + \cdots + \frac{\partial \phi^n_k}{\partial x^n} = 0, \qquad (k=1,\ldots,n)$$

where

$$(0.6) \qquad \phi^i_k = \delta^i_k \, f - \sum_{j=1}^{p} u^j_k \, f_{u^j_i} \, .$$

Unfortunately, the system (0.5) is not necessarily linearly equivalent to (0.4), for example when $n < p$.

Occasionally well-known systems of equations which are not customarily derived from variational principles can be written in the form of a system of conservation laws. For example, the equations

$$(0.7) \qquad \begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ u_t + uu_x + \frac{1}{\rho} p_x = 0 \\ p_t + up_x + \gamma p u_x = 0 \end{cases}$$

of the one-dimensional, non-isentropic, non-steady flow of an adiabatic fluid, where ρ is the density, u the velocity, p the pressure, and γ the adiabatic constant, can be written in the form

$$(0.8) \quad \begin{cases} \operatorname*{div}_{\rho} \ (1, u) = 0 \\ \operatorname*{div}_{\rho} \ (u, u^2 + \frac{p}{\rho}) = 0 \\ \operatorname*{div}_{\rho} \ (u^2 + \frac{2}{\gamma - 1} \frac{p}{\rho}, u^3 + \frac{2\gamma}{\gamma - 1} u \frac{p}{\rho}) = 0. \end{cases}$$

If an arbitrary system of first order quasi-linear homogeneous partial differential equations is given, one might ask if it is equivalent in some sense to a system of conservation laws, or at least how many conservation laws can be obtained from it by linear combination, as in the preceding examples. This question will be stated more specifically in Chapter I and answered for a small class of equations in Chapter II; it is answered in a very general way in Chapter III.

Chapter I is of an algebraic nature, and reduces the problem of the existence of conservation laws to that of solving an over-determined system of linear homogeneous partial differential equations. An existence theorem is given for the over-determined system in Chapter II, under the restriction that certain integrability conditions are satisfied. For the more general problem solved in Chapter III the question is more conveniently stated in terms of exterior differential forms in order that the existence theorem of Cartan and Kähler can easily be applied. A brief sketch of the theory of systems of exterior differential forms has been included for convenience.

The principle results of this investigation suggest that in general one should expect systems of partial differential equations to contain many more conservation laws than appear on the surface. For example, when $n = 2$ and $p = 3$ every system in which the coefficients depend exclusively on the unknowns possesses at least a two-parameter family of conservation laws. Even more surprising, for a wide variety of cases one can find conservation laws which depend not only on a small number of parameters, but also on a certain number of functions of a single variable; in particular, when the integrability conditions of Chapter II are satisfied, one may assign p arbitrary functions of one variable to obtain infinitely many equivalent systems of conservation laws. In other cases, although certain functions again may be assigned arbitrarily, equivalent systems of conservation laws do not exist.

* * *

All definitions will be indicated by underlining new terms as they arise, with an accompanying explanation in the text.

The Einstein summation convention is occasionally used. The dummy indices will always be lower case Greek letters and the range of summation 1,...,p. There is no summation over Latin indices except when indicated by $\sum$.

CHAPTER I

DEFINITIONS AND STATEMENT OF THE PROBLEM

§1. Well-determined systems

This paper deals with formal properties of a system of quasi-linear homogeneous first order partial differential equations

$$(1.1)\qquad \sum_{i,j=1}^{n,p} \alpha^{ki}_{j} \frac{\partial u^j}{\partial x^i} = 0, \qquad k = 1,\dots,m$$

in the independent variables $x = \{x^i\}$ and unknowns $u = \{u^j\}$, where $\alpha^{ki}_{j} = \alpha^{ki}_{j}(x,u)$.

A system is determined if $m = p$ and equations (1.1) are linearly independent at any given point (x_o,u_o) in the product space $X \times U$ of the spaces of x and u. Clearly the determination of a system is independent of the coordinates of U since a change of coordinates $u = u(v)$ merely turns (1.1) into a system

$$(1.2)\qquad \sum_{i,\ell=1}^{n,p} \beta^{ki}_{\ell} \frac{\partial v^\ell}{\partial x^i} = 0, \qquad k = 1,\dots,m$$

where $\beta^{ki}_{\ell} = \sum_{j=1}^{p} \alpha^{ki}_{j} \frac{\partial u^j}{\partial v^\ell}$; since the matrix $\left(\frac{\partial u^j}{\partial v^\ell}\right)$ is non-singular, the rank of the $np \times p$ matrix (β^{ki}_{ℓ}) is the same as that of (α^{ki}_{j}).

Determined systems exist for which the corresponding Cauchy problem would be rather unnatural. For example, if $n = p = 2$ the system

$$(1.3) \qquad \begin{cases} \dfrac{\partial u^1}{\partial x^1} = 0 \\[2ex] \dfrac{\partial u^1}{\partial x^2} = 0 \end{cases}$$

is determined. Its solution, u^1 an arbitrary constant and u^2 an arbitrary function $f(x^1,x^2)$, is uniquely determined by the unusual Cauchy data u^1 = constant and $u^2 = f(x^1,x^2)$. Similarly the system

$$(1.4) \qquad \begin{cases} \dfrac{\partial u^1}{\partial x^1} = 0 \\[2ex] \dfrac{\partial u^2}{\partial x^1} = 0 \end{cases}$$

can be completely solved merely by assigning the Cauchy data $u^1 = f^1(x^2)$, $u^2 = f^2(x^2)$, where f^1 and f^2 are arbitrary.

Let $\xi = \{\xi_i\}$ be indeterminates, and for each i let (α^{ki}_j) be a $p \times p$ matrix with complex entries, $(i = 1,\ldots,n)$. Then pathological examples like (1.3) and (1.4) may be eliminated by requiring that

$$(1.5) \qquad \det\left(\sum_{i=1}^{n} \xi_i \, \alpha^{ki}_j\right) \neq 0$$

be a non-zero form of degree p in ξ. Any system (1.1) satisfying (1.5) is called <u>well-determined</u>. Clearly this property is independent of the coordinates in X as well as in U.

The ξ_i may be assigned complex values which vary covariantly with coordinate changes in X. For any point (x_o, u_o), the annihilator of any co-vector (ξ_i) for which the determinant (1.5) vanishes is called a characteristic element in X at (x_o, u_o). Thus a system is well-determined if and only if not every element is characteristic.

If (1.5) contains a linear factor, the coefficient of ξ_i appearing in this factor may be taken as the i^{th} direction number of a characteristic direction in X at (x_o, u_o). Geometrically, if there exists an (n-2) - parameter family of characteristic elements in X at (x_o, u_o) which have a direction in common, then the direction is characteristic. For $n = 2$, when p linear factors always exist, the preceding definitions coincide, and furthermore, for any characteristic direction one can find a linear combination of the equations of the system of partial differential equations such that every unknown appearing in it is differentiated in the given characteristic direction. For $n > 2$ such linear combinations do not necessarily exist.

§2. Tangent spaces, equivalence of well-determined systems

Let $\{u\} = \{u : X\}$ be a class of differentiable functions u of X. By choosing a set $\{u_1, \ldots, u_q\}$ of functionally independent u's about any given point of X this class becomes a differentiable manifold, also denoted by $\{u\}$, with the values of $u_1, \ldots, u_q$ as the local coordinates. We note that the operator

$\frac{\partial}{\partial x^i} \equiv \sum_{j=1}^{q} \frac{\partial u_j}{\partial x^i} \frac{\partial}{\partial u_j}$ is a linear differential mapping of the class of all differentiable functions on the manifold $\{u\}$ into the complex number field, at any given point of X. Since any operation $\sum_{i=1}^{n} \mu^i \frac{\partial}{\partial x^i}$ on $\{u\}$ is also a linear differential mapping, these mappings form a vector space, <u>the tangent space of $\{u\}$</u>, spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. Two vectors in this space are identical if and only if they furnish the same mapping. Clearly the space is of dimension q.

An alternate definition can be given merely by replacing "the class of all differentiable functions on the manifold $\{u\}$" by "the class $\{u\}$" alone, or even by $\{u_1, \ldots, u_q\}$ and considering the operators $\frac{\partial}{\partial x^i}$. This definition has the advantage that it defines a tangent space to the <u>class</u> $\{u\}$ without constructing the corresponding manifold. As an example, consider the <u>class</u> $\{u\}$ of all solutions u of $\alpha^i \frac{\partial u}{\partial x^i} = 0$. Its tangent space is just the annihilator at any point of the vector $(\alpha^1, \ldots, \alpha^n)$; clearly there is no point in constructing the <u>manifold</u> $\{u\}$ to discover this.

The tangent space just defined is distinct from the <u>tangent space of U</u>, which is a p-dimensional space spanned by the mappings $\frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^p}$ of <u>all</u> differentiable functions on the manifold U, independently of X.

Let $\{(u^1, \ldots, u^p)\} = \{(u^1, \ldots, u^p) : X\}$ be a class of p-tuples of functions u^j of X. Suppose there are q p-tuples $(u^1_1, \ldots, u^p_1), \ldots, (u^1_q, \ldots, u^p_q)$ such that the rank of the $q \times np$ matrix

$$\begin{pmatrix} \frac{\partial u_1^1}{\partial x^1} \cdots \frac{\partial u_1^p}{\partial x^1} \cdots & \cdots \frac{\partial u_1^1}{\partial x^n} \cdots \frac{\partial u_1^p}{\partial x^n} \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \frac{\partial u_q^1}{\partial x^1} \cdots \frac{\partial u_q^p}{\partial x^1} \cdots & \cdots \frac{\partial u_q^1}{\partial x^n} \cdots \frac{\partial u_q^p}{\partial x^n} \end{pmatrix}$$

is q; then the q p-tuples are defined to be <u>independent</u>. Any operator $\left(\sum_{i=1}^{n} \mu_1^i \frac{\partial}{\partial x^i}, \ldots, \sum_{i=1}^{n} \mu_p^i \frac{\partial}{\partial x^i} \right)$ is a linear differential mapping of the class $\{(u^1,\ldots,u^p)\}$ into p-tuples of complex numbers, at any point of X, where the μ's are arbitrary. As before, if there exist exactly q independent p-tuples these maps form a q-dimensional vector space, the <u>tangent space of $\{(u^1,\ldots,u^p)\}$</u>.

Suppose that (1.1) is determined and continuous in $X \times U$ in a neighborhood of (x_o,u_o). Then the p vectors $\alpha^k = (\alpha_1^{k1},\ldots,\alpha_p^{k1},\ldots,\alpha_1^{kn},\ldots,\alpha_p^{kn})$, $k = 1,\ldots,p$ are independent in the usual sense for each (x,u) in the neighborhood and span a p-dimensional subspace V of some fixed np-dimensional complex vector space.

Assume that for any well-determined system

$$(1.6) \qquad \sum_{i=1}^{n} \alpha_\rho^{ki} \frac{\partial u^\rho}{\partial x^i} = 0, \quad k = 1,\ldots,p$$

the tangent space T of the class of solutions $\{(u^1,\ldots,u^p)\}$ of (1.6) is isomorphic to V*, the dual of V. Then one can define two systems to be equivalent if the tangent spaces of their classes of solutions are isomorphic to each other. Clearly this implies that they possess the same solutions.

Without using the above assumption, two systems (1.6) will be called equivalent if the corresponding spaces V* are isomorphic. This means simply that one can apply a non-singular linear transformation, depending on $X \times U$, to obtain the equations of one system from those of the other. This definition is less satisfactory than the one just suggested only in that it characterizes V* purely formally.

§3. Statement of the problem

If a given system is equivalent to a system one of whose equations happens to be a conservation law, the original system is said to contain the conservation law. Geometrically, the space dual to the vector consisting of the coefficients of the $\frac{\partial u^j}{\partial x^i}$ appearing in the conservation law contains V*. As an example suppose $n = 2$ and $p = 1$. Then if α^1 and α^2 are differentiable functions of u alone, the system

$$\alpha^1 \frac{\partial u}{\partial x^1} + \alpha^2 \frac{\partial u}{\partial x^2} = 0 \tag{1.7}$$

contains infinitely many conservation laws, depending on a single arbitrary function. Namely,

$$\frac{\partial}{\partial x^1} \int^u \alpha^1(v)f(v)dv + \frac{\partial}{\partial x^2} \int^u \alpha^2(v)f(v)\,dv = 0, \tag{1.8}$$

where $f = f(v)$ is the arbitrary function. Note that the tentative assumption of §2 always holds for $p = 1$ so that the equivalence of (1.7) and (1.8) could be taken in the earlier sense if desired.

The problem is to find how many conservation laws are contained in a given system. In particular it is of interest to learn when a system is equivalent to another system consisting entirely of conservation laws, (0.8) for example, and in how many ways such a representation can be found. One surprising result is that even if those representations which can be obtained from one another by linear combination with constant coefficients are included in the same equivalence class, there is still a wide variety of systems which can be represented by infinitely many equivalence classes.

From now on we restrict ourselves to two independent variables, $n = 2$, except as noted. The number of unknowns will be arbitrary, although all of the chief difficulties of the problem occur when $p = 3$. We further assume that the functions α_j^{ki} are independent of X. Thus the existence of conservation laws is a purely local problem in U. The functions α_j^{ki} are assumed to be as many times differentiable as needed in some neighborhood of $u = u$; they are assumed to be analytic in Chapter III. Only well-determined systems will be considered. Finally, in order to avoid considering several exceptional cases, the p characteristic directions in X will be assumed to depend continuously on U and to be distinct, but not necessarily real, at (x_o, u_o), except as noted. Thus it does not matter whether the system is totally hyperbolic or not, e.g.

For convenience, specific reference to u_o or (x_o, u_o) is usually omitted. All the results of this paper will be given only for a neighborhood of this point. If the appropriate conditions are satisfied in a larger portion of U or of $X \times U$, the results can clearly be extended to this portion.

§4. The corresponding linear equations

Let the system

$$(1.9) \quad \alpha^{k1}_{\rho} \frac{\partial u^{\rho}}{\partial x^1} + \alpha^{k2}_{\rho} \frac{\partial u^{\rho}}{\partial x^2} = 0, \quad k = 1,\dots,p$$

satisfy the assumptions of §3 and let $\xi^{\ell} = (\xi^{\ell}_1, \xi^{\ell}_2)$ annihilate the ℓ^{th} characteristic direction in X, $\ell = 1,\dots,p$. The matrix $(\xi^{\ell}_1 \alpha^{k1}_j + \xi^{\ell}_2 \alpha^{k2}_j)$ is of rank p-1, so that to each characteristic direction in X there corresponds a unique <u>characteristic direction in U</u>, denoted by $\eta_{\ell} = (\eta^1_{\ell},\dots,\eta^p_{\ell})$ which is defined by

$$(1.10) \quad (\xi^{\ell}_1 \alpha^{k1}_{\rho} + \xi^{\ell}_2 \alpha^{k2}_{\rho})\eta^{\rho}_{\ell} = 0, \quad (\text{no summation on } \ell).$$

Since the characteristic directions in X are distinct the matrix (η^{j}_{ℓ}) is non-singular. Setting

$$(1.11) \quad \beta^{ki}_{\ell} = \alpha^{ki}_{\rho}\eta^{\rho}_{\ell}$$

equation (1.10) becomes

$$(1.13) \quad \xi^{\ell}_1 \beta^{k1}_{\ell} + \xi^{\ell}_2 \beta^{k2}_{\ell} = 0 \qquad (k = 1,\dots,p) \quad (\text{no summation on } \ell),$$

so that the ratio $\beta^{k1}_{\ell} : \beta^{k2}_{\ell}$ is independent of k. Furthermore, for at least one value of k not both β^{k1}_{ℓ} and β^{k2}_{ℓ} vanish since (1.9) is well-determined, that is, (1.14) $\det \; (\xi_1 \beta_\ell^{k1} + \xi_2 \beta_\ell^{k2}) \neq 0 \, .$

Let $(\bar{u}_1^1, \ldots, \bar{u}_1^p; \bar{u}_2^1, \ldots, \bar{u}_2^p)$ be any member of V* and set

(1.15) $\bar{u}_i^j = \eta_\rho^j \, \bar{\sigma}_i^\rho$

so that in the new coordinates V* is defined by

(1.16) $\beta_\rho^{k1} \bar{\sigma}_1^\rho + \beta_\rho^{k2} \bar{\sigma}_2^\rho = 0, \qquad k = 1, \ldots, p.$

Since $\beta_\ell^{k1} : \beta_\ell^{k2}$ is well-defined and independent of k we may use the elements

$\sigma^\ell = (0, \ldots, \bar{\sigma}_1^\ell, \ldots, 0; 0, \ldots, \bar{\sigma}_2^\ell, \ldots, 0)$ as a basis in V* with respect to the new coordinates, where

(1.17) $\bar{\sigma}_1^\ell \beta_\ell^{k1} + \bar{\sigma}_2^\ell \beta_\ell^{k2} = 0$, (no summation on ℓ).

But (1.13) and (1.17) imply that

(1.18) $\det \begin{pmatrix} \xi_1^\ell & \xi_2^\ell \\ \bar{\sigma}_1^\ell & \bar{\sigma}_2^\ell \end{pmatrix} = 0, \qquad \ell = 1, \ldots, p,$

which is merely the definition of V* in the new coordinates.

Suppose (1.19) contains a conservation law

(1.19) $\dfrac{\partial \phi^1}{\partial x^1} + \dfrac{\partial \phi^2}{\partial x^2} = \dfrac{\partial \phi^1}{\partial u^\rho} \dfrac{\partial u^\rho}{\partial x^1} + \dfrac{\partial \phi^2}{\partial u^\rho} \dfrac{\partial u^\rho}{\partial x^2} = 0 \, .$

Setting

$$(1.20) \qquad U_\ell \phi^i = \frac{\partial \phi^i}{\partial u^\rho} \eta^\rho_\ell \,, \quad \frac{\partial u^j}{\partial x^i} = \eta^j_\rho \sigma^\rho_i$$

this may be written as

$$(1.21) \qquad \sigma^\rho_1 U_\rho \phi^1 + \sigma^\rho_2 U_\rho \phi^2 = 0 \,,$$

and we may speak of the <u>tangent vectors</u> U_ρ in the tangent space of U. Then the elements which are dual to a space consisting of the single vector $(U_1\phi^1,\ldots,U_p\phi^1;\ U_1\phi^2,\ldots,U_p\phi^2)$ must contain V*, expressed in the new coordinates. It will suffice to look at the basis elements of V*. For each ℓ we have

$$(1.22) \qquad \bar{\sigma}^\ell_1 U_\ell \phi^1 + \bar{\sigma}^\ell_2 U_\ell \phi^2 = 0, \quad (\text{no summation on } \ell,\ \ell = 1,\ldots,p).$$

Referring to (1.18) this becomes

$$(1.23) \qquad \xi^\ell_1 U_\ell \phi^1 + \xi^\ell_2 U \phi^2 = 0, \quad (\text{no summation on } \ell,\ \ell = 1,\ldots,p).$$

It should be noted that these equations are entirely independent of X. <u>Except for the trivial constant solutions, there corresponds a conservation law to every solution of (1.23), which is a linear system of partial differential equations in the space U.</u>

§5. The case $p = 2$

In this case the tangent vectors in U are of the form

$$(1.24) \qquad U_j = \eta_j^1 \frac{\partial}{\partial u^1} + \eta_j^2 \frac{\partial}{\partial u^2}, \qquad j = 1,2 .$$

Let (ζ_k^j) be the inverse of (η_i^k), $\zeta_1^j \eta_i^1 + \zeta_2^j \eta_i^2 = \delta_i^j$.

Then an integrating factor M_j always exists such that

$M_j(\zeta_1^j \, du^1 + \zeta_2^j \, du^2)$ is a total differential, say dv^j, $j = 1,2,$

so that

$$(1.25) \qquad M_j \zeta_k^j = \frac{\partial v^j}{\partial u^k}, \qquad \text{(no summation)};$$

hence

$$(1.26) \qquad \eta_j^k = M_j \frac{\partial u^k}{\partial v^j},$$

and (1.23) becomes

$$(1.27) \qquad \zeta_1^j \frac{\partial \phi^1}{\partial v^j} + \zeta_2^j \frac{\partial \phi^2}{\partial v^j} = 0, \qquad j = 1,2, \text{ after division by } M_j.$$

The system (1.27) has a number of solutions about u_o defined by two arbitrary functions of a single variable, as one easily shows by the Cauchy-Kowalewski theorem, if the ζ_i^j are assumed to be analytic. This existence theorem will be given in a particularly nice form as a special case of a more general result in Chapter III, §§3, 5. Thus for $p = 2$ (1.9) is equivalent to many systems of conservation laws [3].

§6. The case $p > 2$

In general one cannot expect to find integrating factors leading to the simple form (1.27) of equations (1.23); the case where this can be done is treated in Chapter II.

In any case the system (1.23), or any system equivalent to it, is over-determined for $p > 2$, that is, there are more linearly independent equations than unknowns. Therefore, in order to find results analogous to those found in §3 and §5 for $p = 1$ and $p = 2$ respectively, one should expect that certain integrability conditions must be satisfied.

As noted in §4 the matrix (η_ℓ^j) is non-singular, hence the p operators U_ℓ are differentiations in p distinct directions which span the tangent space in any point of U. The U_ℓ do not commute with each other in general. However, their commutators

$$(1.28)\qquad [U_\ell\, U_m] \equiv \sum_{j,k=1}^{p,p} \left(\eta_\ell^j \frac{\partial \eta_m^k}{\partial u^j} - \eta_m^j \frac{\partial \eta_\ell^k}{\partial u^j}\right) \frac{\partial}{\partial u^k}$$

again lie in the tangent space and so may be expressed in the form

$$(1.29)\qquad [U_\ell\, U_m] \equiv \sum_{j=1}^{p} \gamma_{\ell m}^j\, U_j$$

for certain analytic functions $\gamma_{\ell m}^j = \gamma_{\ell m}^j(u)$ called the structure functions. One easily checks that for any suitably differentiable function $\phi = \phi(u)$

$$(1.30)\qquad U_\ell(U_m\phi) - U_m(U_\ell\phi) = [U_\ell U_m]\,\phi\ .$$

Clearly

(1.31) $\quad \gamma^{j}_{\ell m} + \gamma^{j}_{m\ell} = 0$;

furthermore the $\gamma^{j}_{\ell m}$'s must satisfy a relation arising from Jacobi's identity

$$\underset{\ell mn}{Z} \; [[U_\ell U_m]U_n] \equiv$$

(1.32) $\quad [[U_\ell U_m]U_n] + [[U_m U_n]U_\ell] + [[U_n U_\ell]U_m] \equiv 0,$

where Z represents cyclic summation as indicated. If the $\gamma^{j}_{\ell m}$'s are constant, the fundamental theorem of Lie groups asserts that there exists an analytic group for which the U_ℓ are the infinitesimal transformations.

If $\gamma^{j}_{\ell m} = 0$ except when $j = \ell$ or $j = m$ one can find integrating factors M_ℓ and a change of variables, $v = v(u)$, so that

(1.33) $\quad U_\ell \equiv M_\ell \frac{\partial}{\partial v} \ell, \qquad \ell = 1,\ldots,p$

as in §5. This will be shown in Chapter II, which deals entirely with this special case, and again by a simpler means in Chapter III.

§7. A preliminary transformation

The elementary methods of Chapter II are better suited to deal with the case where either (α^{k1}_{j}) or (α^{k2}_{j}) is a non-singular matrix. Suppose that (α^{k1}_{j}) is singular. Since (1.9) is well-determined there exist constants $\mu^i = (\mu^i_1, \mu^i_2)$ such that $(\mu^1_1 \alpha^{k1}_j + \mu^1_2 \alpha^{k2}_j)$ is non-singular in a neighborhood of $u = u_o$ and such that μ^1 and μ^2 are distinct directions in the dual of the tangent space of X. Letting

$$(1.34) \qquad \bar{x}^i = \mu^i_1 x^1 + \mu^i_2 x^2 , \qquad i = 1,2$$

so that

$$(1.35) \qquad \frac{\partial}{\partial x^i} = \mu^1_i \frac{\partial}{\partial \bar{x}^1} + \mu^2_i \frac{\partial}{\bar{x}^2} ,$$

we see that (1.9) may be written

$$(1.36) \qquad \sum_{j=1}^{p} (\bar{\alpha}^{k1}_j \frac{\partial u^j}{\partial \bar{x}^1} + \bar{\alpha}^{k2}_j \frac{\partial u^j}{\partial \bar{x}^2}) = 0$$

where

$$(1.37) \qquad \bar{\alpha}^{ki}_j = \mu^i_1 \alpha^{k1}_j + \mu^i_2 \alpha^{k2}_j$$

and $(\bar{\alpha}^{k1}_j)$ is non-singular. Thus (α^{k1}_j) might as well be assumed non-singular at the outset.

If

$$(1.38) \qquad \frac{\partial u^j}{\partial \bar{x}^i} = \sum_{\ell=1}^{p} \eta^j_\ell \, \bar{\sigma}^\ell_i$$

then

$$(1.39) \qquad \sigma^\ell_i = \mu^1_i \bar{\sigma}^\ell_1 + \mu^2_i \bar{\sigma}^\ell_2 .$$

But the form

$$(1.40)\qquad \sigma_1^{\ell}\, U_{\ell}\, \bar{\phi}^1 + \sigma_1^{\ell}\, U_{\ell}\bar{\phi}^2 = 0$$

of a conservation law is invariant with respect to linear coordinate changes in X. Hence the existence conditions (1.23) become

$$(1.41)\qquad \bar{\xi}_1^{\ell}\, U_{\ell}\bar{\phi}^1 + \bar{\xi}_2^{\ell}\, U_{\ell}\bar{\phi}^2 = 0$$

where $\xi_i^{\ell} = \mu_i^1\; \bar{\xi}_1^{\ell} + \mu_i^2\; \bar{\xi}_2^{\ell}$. Thus the ratios $\xi_1^{\ell} : \xi_2^{\ell}$ at any point of U may be changed by a projective transformation. In particular, since the projective group is simply transitive on any three of these ratios, for $p = 3$ one may adjust the characteristic directions in X to be any three arbitrary distinct directions at u_o. This transformation could have been derived directly from (1.23) simply by setting

$$(1.42)\qquad \bar{\phi}^i = \mu_1^i\, \phi^1 + \mu_2^i\, \phi^2$$

without investigating the corresponding transformation in X.

Taking (α_j^{kl}) to be non-singular merely corresponds to $\xi_1^{\ell} \neq 0$, $\ell = 1,\ldots,p$.

Assuming that (α_j^{kl}) is non-singular, equations (1.9) can be written in the unsymmetric form

$$(1.43)\qquad \frac{\partial u^k}{\partial x^1} + \sum_{j=1}^{p} \alpha_j^k\, \frac{\partial u^j}{\partial x^2} = 0, \qquad k = 1,\ldots,p.$$

In this case the derivation of § 4 is easier to give. For suppose a conservation law exists, so that

$$(1.19) \quad \sum_{j=1}^{p} \left(\frac{\partial \phi^1}{\partial u^j} \frac{\partial u^j}{\partial x^1} + \frac{\partial \phi^2}{\partial u^j} \frac{\partial u^j}{\partial x^2} \right) = 0.$$

Multiplying (1.43) by $\frac{\partial \phi^1}{\partial u^k}$ and adding,

$$(1.44) \quad \sum_{k=1}^{p} \frac{\partial \phi^1}{\partial u^k} \frac{\partial u^k}{\partial x^1} + \sum_{j,k=1}^{p,p} \frac{\partial \phi^1}{\partial u^k} \alpha^k_j \frac{\partial u^j}{\partial x^2} = 0,$$

hence

$$(1.45) \quad \sum_{j,k=1}^{p,p} \frac{\partial \phi^1}{\partial u^k} \alpha^k_j \frac{\partial u^j}{\partial x^2} = \sum_{j=1}^{p} \frac{\partial \phi^2}{\partial u^j} \frac{\partial u^j}{\partial x^2}.$$

But the p elements $\frac{\partial u^j}{\partial x^2}$, $j = 1,\ldots,p$ determine a basis in V*, so that (1.45) implies

$$(1.41) \quad \sum_{k=1}^{p} \frac{\partial \phi^1}{\partial u^k} \alpha^k_j = \frac{\partial \phi^2}{\partial u^j}, \qquad j = 1,\ldots,p$$

from which one derives (1.23) merely by diagonalizing (α^k_j), which is possible because the characteristic directions in X are distinct.

CHAPTER II

THE SPECIAL CASE

§1. The tangent vectors U_j

A set of tangent vectors (operators) U_j, $j = 1,\dots,q$ is complete if and only if the commutator of any pair of them lies in the tangent space the set spans. (For convenience here we speak of tangent vectors even in the absence of a manifold or class on which a tangent vector could be defined.) It is well-known that for any complete set, depending continuously on r parameters, $r > q$, then there exists an $r - q$ parameter family of integral manifolds in the parameter space whose tangent spaces are spanned by the given vectors.

Given a set of vectors U_j, $j = 1,\dots,p$, suppose there exist non-zero factors M_j and a change of variables, $v = v(u)$, such that these vectors are of the form

$$(2.1) \qquad U_j \equiv M_j \frac{\partial}{\partial v^j}, \qquad j = 1,\dots,p.$$

Then

$$(2.2) \qquad [U_j U_i] \equiv \gamma^i_{ji} U_i - \gamma^i_{ji} U_j$$

where

$$(2.3) \qquad M_i \gamma^i_{ji} = U_j M_i,$$

all the other structure functions vanishing identically. This is clearly equivalent to the assertion that any subset of the vectors U_j, $j = 1,\dots,p$, is complete. Conversely, the following theorem holds.

THEOREM A: If any subset of the vectors U_j, $j = 1,\dots,p$ is complete, then there exist non-zero factors M_j and a change of variables, $v = v(u)$, satisfying (2.1).

Proof: A sequence of changes of variables will be constructed in such a way that at the n^{th} stage the form (2.1) is displayed for all j, $j = 1,\dots,n$ $n \leq p$. Assume that the first $n - 1$ stages have been completed. Then, after multiplication by suitable scalers, the first n vectors may be assumed to be of the form $\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^{n-1}}$, $\eta^\rho \frac{\partial}{\partial u^\rho}$, where $\eta^n = 1$. Since by assumption the commutator $[\frac{\partial}{\partial u^j}, \eta^\rho\frac{\partial}{\partial u^\rho}]$ is a combination of $\frac{\partial}{\partial u^j}$ and $\eta^\rho \frac{\partial}{\partial u^\rho}$, we have $\frac{\partial}{\partial u^j}\eta^k = \lambda_j \eta^k$ for some λ_j, $j \neq k$. But $\eta^n = 1$ implies $\frac{\partial \eta^n}{\partial u^j} = 0$, hence $\lambda_j = 0$, so that $\eta^j = \eta^j(u^j; u^n,\dots,u^p)$ for $j = 1,\dots,n-1$, and $\eta^j = \eta^j(u^n,\dots,u^p)$ for $j = n,\dots,p$.

The n^{th} change of variables will be assumed to have the form

$$(2.4)\qquad \begin{cases} v^j = v^j(u^j; u^n,\dots,u^p) & j = 1,\dots,n-1 \\ v^j = v^j(u^n,\dots,u^p) & j = n,\dots,p. \end{cases}$$

Then for $j = 1,\dots,n-1$

$$(2.5)\qquad U_j \equiv \frac{\partial}{\partial u^j} \equiv \frac{\partial v^\rho}{\partial u^j}\frac{\partial}{\partial v^\rho} \equiv \frac{\partial v^j}{\partial u^j}\ \frac{\partial}{\partial v^j}, \qquad \text{(no summation on } j)$$

so that the first $n - 1$ vectors retain the desired form provided

$\frac{\partial v^j}{\partial u^j} \neq 0$. The n^{th} vector becomes

$$\text{(2.6)} \qquad U_n \equiv \eta^\rho \frac{\partial}{\partial u^\rho} \equiv \eta^\rho \frac{\partial v^\sigma}{\partial u^\rho} \frac{\partial}{\partial v^\sigma},$$

and hence will have the form $U_n \equiv \eta^\rho \frac{\partial v^n}{\partial u^\rho} \frac{\partial}{\partial v^n}$ if and only if $\eta^\rho \frac{\partial v^j}{\partial u^\rho}$ vanishes for $j \neq n$. For $j = 1,\ldots,n-1$ this requirement is

$$\text{(2.7)} \qquad \eta^j \frac{\partial v^j}{\partial u^j} + \sum_{k=1}^{p} \eta^k \frac{\partial v^j}{\partial u^k} = 0, \qquad j = 1,\ldots,n-1,$$

and for each of these equations one can find a v^j for which $\frac{\partial v^j}{\partial u^j} \neq 0$. To satisfy the requirement for $j = n,\ldots,p$, each of the v^j's, $j = n,\ldots,p$, may be chosen as an <u>independent</u> solution of the equation

$$\text{(2.8)} \qquad \sum_{k=n}^{p} \eta^k \frac{\partial v}{\partial u^k} = 0,$$

in which the coefficients depend only on $u^n,\ldots,u^p$. In particular, v^n may be chosen in such a way that $\frac{\partial v^n}{\partial u^n} \neq 0$. Clearly the Jacobian matrix of the transformation (2.4) chosen in this way is non-singular. For there is nothing below the first $n - 1$ terms on the principal diagonal, and the block of side $p - n + 1$ in the lower right-hand corner represents the non-singular change of variables chosen last. This completes the proof.

Some further properties of vectors satisfying (2.1) will be given for convenience. First, for any differentiable function f, (2.3) implies

$$U_j\left(\frac{f}{M_i}\right) = \frac{U_j f}{M_i} - f\,\frac{U_j M_i}{(M_i)^2}$$

(2.9)

$$= \frac{1}{M_i}\left(U_j f + f\,\gamma^i_{ij}\right) = \frac{1}{M_i}\,U_{j;i} f$$

where $U_{j;k}$ is the operator $M_k U_j\left(\frac{\ }{M_k}\right) = U_j + \gamma^i_{ij}$.

If $f_j = \frac{\partial f}{\partial v^j}$, $j = 1,\dots,p$, then

$$0 = \frac{\partial f_j}{\partial v^i} - \frac{\partial f_i}{\partial v^j} = \frac{1}{M_i}\,U_i f_j - \frac{1}{M_j}\,U_j f_i$$

(2.10)

$$= \frac{1}{M_i M_j}\left\{U_{i;j}\,(M_j f_j) - U_{j;i}\,(M_i f_i)\right\}.$$

In particular, since (2.3) can be written

(2.11) $$\gamma^k_{jk} = M_j \frac{\partial}{\partial v^j} \log M_k ,$$

we have

(2.12) $$U_{i;j}\,\gamma^k_{jk} = U_{j;i}\,\gamma^k_{ik},\ i,j,k = 1,\dots,p.$$

Equation (2.10) can be given an interpretation in terms of the operators U_j alone. Given p functions g_j, a necessary and sufficient condition that there exist a single function g such that $g_j = U_j g$, $j = 1,\dots,p$ is that

(2.13) $$U_{j;i} g_i = U_{i;j} g_j, \qquad i,j = 1,\dots,p.$$

Another proof of this statement follows from a form of Stokes' theorem. Let $\underset{ijk}{Z}$ represent cyclic summation, $\underset{ijk}{Z} F(ijk) = F(ijk) + F(kij) + F(jki)$, and suppose $\eta^j_\sigma du^\sigma = \omega^j$, then for $p = 3$ we have

$$\int_{\partial R} dg = \int_{\partial R} \beta_\rho \, \omega^\rho$$

(2.14)

$$= \int_R \underset{\lambda\mu\nu}{Z} (U_{\lambda;\mu}\beta_\mu - U_{\mu;\lambda}\beta_\lambda) \underset{ijk}{Z} \frac{\eta^i_\nu du^j du^k}{E}$$

where $E = \det \eta^j_k$. Formula (2.14) can be proved by means of the more customary form of the Stokes theorem.

The factors M_j can be calculated directly from (2.3) without involving the sequence of changes of variable used in the proof of THEOREM A. Once these factors are known the desired change of variables can be made directly.

An alternate proof of THEOREM A will be sketched, based on the direct computation just suggested. By performing the calculation indicated in Jacobi's identity, (1.32), substituting $\gamma^\rho_{jk} U_\rho$ for $[U_j U_k]$ whenever possible, one finds (2.12) under the assumption that $\gamma^k_{ij} = 0$ except for $k = j$ or $i = k$. For each i the system (2.3) can be converted into a homogeneous system by considering M_i as a new independent variable; then M_i can be found in terms of the other independent variables, if the new system is complete, simply by considering the constant solutions to the new system. It turns out that (2.12) is exactly the condition that the homogeneous system is complete, hence the factors M_i exist. Thus the commutators of the vectors $\frac{1}{M_i} U_i$ all vanish, from which it follows that they are of the desired form, $\frac{\partial}{\partial v^i}$.

A third proof of THEOREM A is given in Chapter III, §2.

In the new coordinates the v^j axis may be described as the solution of

$$(2.15) \qquad \frac{dv^1}{0} = \ldots = \frac{dv^j}{1} = \ldots = \frac{dv^p}{0}$$

passing through the origin. In the original coordinates this is the solution of

$$(2.16) \qquad \frac{du^1}{\eta^1_j} = \ldots = \frac{du^p}{\eta^p_j}$$

passing through u_o. The curves thus obtained for $j = 1,\ldots,p$ are at each point tangent to a characteristic direction in U for the system (1.9) and are called the *characteristic curves* through u_o. As THEOREM A has shown, it is the peculiarity of the special case being considered in this chapter, in which any subset of the tangent vectors U_j is assumed to be complete, that the p families K_j of characteristic curves through all points of U may be used to define a *coordinate system*. This means that if $j_1,\ldots,j_p$ is any permutation of $1,\ldots,p$ then those members of K_{j_2} intersecting given members of K_{j_1} form the same family $K_{j_1 j_2}$ of two-dimensional surfaces when j_1 and j_2 are interchanged; those members of K_{j_3} intersecting given members of $K_{j_1 j_2}$ form the same family of three-dimensional surfaces when $j_1\ j_2\ j_3$ are permuted, etc. These hypersurfaces are called the webs spanned by $j_1\ j_2$, $j_1\ j_2\ j_3$, etc.

§2. Conditions for a representation by a system of conservation laws

Assume that an appropriate change of variables has been performed as in Chapter I, §7, so that (α^{kl}_{j}) is non-singular. Then $\xi^j_1 \neq 0$, $j = 1,\ldots,p$ so that in the special case, in which any subset of the tangent vectors U_j is complete, the existence conditions (1.23) become

$$(2.17)\qquad \frac{\partial \phi^1}{\partial v^j} + \xi^j \frac{\partial \phi^2}{\partial v^j} = 0, \qquad j = 1,\ldots,p$$

after division by M_j, where $\xi^j_2 = \xi^j_1\, \xi^j$. Note that the ξ^j are distinct. The number of solutions of (2.17) gives the number of conservation laws contained in (1.9). More specifically, the question here is to represent (1.9) by a system of conservation laws. The necessary and sufficient condition is that there exist solutions $\left(\frac{\partial\phi^1}{\partial v^1},\ldots,\frac{\partial\phi^1}{\partial v^p}; \frac{\partial\phi^2}{\partial v^1},\ldots,\frac{\partial\phi^2}{\partial v^p}\right)$ of (2.17) which span V. Hence, from (2.17), where the values of $\frac{\partial\phi^2}{\partial v^1},\ldots,\frac{\partial\phi^2}{\partial v^p}$ determine those of $\frac{\partial\phi^1}{\partial v^1},\ldots,\frac{\partial\phi^1}{\partial v^p}$, it is clear that a necessary condition is that no linear relation exist among the $\frac{\partial\phi^2}{\partial v^1},\ldots,\frac{\partial\phi^2}{\partial v^p}$.

We try to concoct such a linear relation. Let i j k represent any three distinct integers among $1,\ldots,p$.

Since $\frac{\partial^2\phi^1}{\partial v^k\, \partial v^j} = \frac{\partial^2\phi^1}{\partial v^j \partial v^k}$ we may cross-differentiate the equations of (2.17) to find

$$(2.18)\quad \frac{\partial \xi^j}{\partial v^k}\frac{\partial \phi^2}{\partial v^j} + \xi^j \frac{\partial^2 \phi^2}{\partial v^k \partial v^j} = \frac{\partial \xi^k}{\partial v^j}\frac{\partial \phi^2}{\partial v^k} + \xi^k \frac{\partial^2 \phi^2}{\partial v^j \partial v^k},$$

that is, setting $\phi^2 = \phi$ for convenience,

$$(2.19)\quad \frac{\partial^2 \phi}{\partial v^j \partial v^k} = \frac{1}{\xi^j - \xi^k}\left(\frac{\partial \xi^k}{\partial v^j}\frac{\partial \phi}{\partial v^k} - \frac{\partial \xi^j}{\partial v^k}\frac{\partial \phi}{\partial v^j}\right),$$

since $\xi^j \neq \xi^k$. It should be noted that the system of $\binom{p}{2}$ second order equations obtained by writing down every instance of (2.19) is entirely equivalent to (2.17). Differentiating (2.19) with respect to v^i gives rise to new mixed second derivatives on the right hand side, which can be evaluated exactly as in (2.19) in terms of the first derivatives of ϕ. Similarly the third mixed derivative $\frac{\partial^3 \phi}{\partial v^j \partial v^i \partial v^k}$ can be expressed in terms of the first derivatives. Equating the two third derivatives gives a linear relation among the first derivatives of ϕ all of whose coefficients must vanish if the vectors $\left(\frac{\partial \phi^1}{\partial v^1},\ldots,\frac{\partial \phi^1}{\partial v^p};\frac{\partial \phi^2}{\partial v^1},\ldots,\frac{\partial \phi^2}{\partial v^p}\right)$ span V. The coefficient of $\frac{\partial \phi}{\partial v^k}$ in this linear relation is especially easy to compute if one notes that the mixed second derivative $\frac{\partial^2 \phi}{\partial v^i \partial v^j} = \frac{\partial^2 \phi}{\partial v^j \partial v^i}$ is expressed as a linear combination of only the first derivatives $\frac{\partial \phi}{\partial v^i}$ and $\frac{\partial \phi}{\partial v^j}$. Equating this coefficient to zero gives

$$(2.20)\quad \frac{\partial}{\partial v^i}\left(\frac{1}{\xi^j - \xi^k}\frac{\partial \xi^k}{\partial v^j}\right) - \frac{\partial}{\partial v^j}\left(\frac{1}{\xi^i - \xi^k}\frac{\partial \xi^k}{\partial v^i}\right) = 0$$

as the necessary condition.

One might expect that the coefficients of $\frac{\partial \phi}{\partial v^i}$ and $\frac{\partial \phi}{\partial v^j}$ would lead to new relations which cannot be obtained merely by permuting the indices $i\ j\ k$ in (2.20). They do not. In fact, letting Θ^k_{ij} represent the left-hand side of (2.20), the entire first order equation in ϕ becomes

$$(2.21) \qquad \underset{ijk}{Z} \frac{\Theta^k_{ij}}{\xi^i - \xi^j} \frac{\partial \phi}{\partial v^k} = 0,$$

where Z represents cyclic summation,

$$\underset{ijk}{Z}\ F(ijk) = F(ijk) + F(jki) + F(kij).$$

Equation (2.21) can be proved by direct calculation, which is not very instructive. A simple reason that one set of indices $i\ j\ k$ leads to only one linear relation among the first derivatives of ϕ will be given in Chapter III, §3.

Equation (2.21), when written back in terms of the original variables and tangent vectors U_j, could be derived directly without making a change of coordinates. This might lead one to expect that a similar derivation with any tangent vectors U_j, not restricted to the special case, would give a similar result. In fact an analogue to (2.21) can be found, but unfortunately it is no longer in general a first order equation; indeed it contains terms $\frac{\gamma^k_{ij}}{\xi^i - \xi^j} U_k (U_k \phi)$, where $i\ j\ k$ are distinct. Thus the only case in which the resulting equation is of first order is the special case considered in this chapter. The more general second order equation can of course be used to derive further results in the general case, however, which will be done in a different setting in Chapter III.

The integrability condition (2.20) for the special case will be more useful expressed in terms of the original variables and tangent vectors, since the coordinate transformation, $v = v(u)$, and the integrating factors M_j might be inconvenient to find in any given example.

Setting $f^k_j = \frac{1}{\xi^j - \xi^k} \frac{\partial \xi^k}{\partial v_j}$, we note that according to (2.10) relation (2.20) may be written as

$$(2.22) \qquad U_{i;j}(M_j f^k_j) = U_{j;i}(M_i f^k_i),$$

that is

$$(2.23) \qquad U_{i;j}\left(\frac{U_j \xi^k}{\xi^j - \xi^k}\right) = U_{j;i}\left(\frac{U_i \xi^k}{\xi^i - \xi^k}\right).$$

Thus we have proved

<u>THEOREM B:</u> Suppose $\alpha^k_j = \alpha^k_j(u)$ is a $p \times p$ matrix of twice differentiable functions of u which has p distinct characteristic roots ξ^k. Then the system

$$(2.24) \qquad \frac{\partial u^k}{\partial x^1} + \alpha^k_\sigma \frac{\partial u^\sigma}{\partial x^2} = 0, \qquad k = 1,\dots,p$$

may be expressed in the form

$$(2.25) \qquad \eta^k_\sigma \frac{\partial u^\sigma}{\partial x^1} + \xi^k \eta^k_\sigma \frac{\partial u^\sigma}{\partial x^2} = 0, \qquad k = 1,\dots,p.$$

Let U_j represent the tangent vector $\eta^\rho_j \frac{\partial}{\partial u^\rho}$, and suppose that any subset of the vectors $U_1, \dots, U_p$ is complete. Then a necessary condition that (2.24) is equivalent to a system of conservation laws is that

$$(2.23) \qquad U_{i;j}\left(\frac{U_j \xi^k}{\xi^j - \xi^k}\right) = U_{j;i}\left(\frac{U_i \xi^k}{\xi^i - \xi^k}\right)$$

for every distinct i j k among 1,...,p. According to (2.13) it is equivalent to demand that there exist p functions f^k such that

$$(2.24) \qquad U_j f^k = \frac{U_j \xi^k}{\xi^j - \xi^k}, \qquad j,k = 1,\dots,p, \quad j \neq k.$$

* * *

Clearly any system with constant coefficients satisfies the conditions of the preceding theorem. To construct a non-trivial system satisfying these conditions one might merely write down a system of conservation laws, attempting to choose p pairs of functions in such a way that the resulting system falls into the special case. This method is much more difficult than attempting to guess the functions ξ^i directly. It is easy to check that the following example satisfies the conditions:

$$(2.25) \qquad \begin{cases} \dfrac{\partial u^1}{\partial x^1} + u^2 u^3 \dfrac{\partial u^1}{\partial x^2} = 0 \\[2ex] \dfrac{\partial u^2}{\partial x^1} + u^3 u^1 \dfrac{\partial u^2}{\partial x^2} = 0 \\[2ex] \dfrac{\partial u^3}{\partial x^1} + u^1 u^2 \dfrac{\partial u^3}{\partial x^2} = 0. \end{cases}$$

Another simple example is that in which $\xi^i = \xi^i(u^i)$, $i = 1,\dots,p$, whose solution one can easily give by assigning p arbitrary functions exactly as one was assigned in (1.8).

§ 3. Proof that the preceding conditions are sufficient

According to THEOREM B there exist functions f^k such that in an appropriate coordinate system

$$(2.26) \qquad \frac{\partial f^k}{\partial v^i} = \frac{1}{\xi^i - \xi^k} \frac{\partial \xi^k}{\partial v^i}, \qquad i,k = 1,\dots,p$$

so that (2.19) might be written

$$(2.27) \qquad \frac{\partial^2 \phi}{\partial v^j \partial v^k} = \frac{\partial f^k}{\partial v^j} \frac{\partial \phi}{\partial v^k} + \frac{\partial f^j}{\partial v^k} \frac{\partial \phi}{\partial v^j}$$

where $\phi = \phi^2$. By cross-differentiation as before it is clear that

$$(2.28) \qquad \frac{\partial^2 f^i}{\partial v^j \partial v^k} + \frac{\partial f^i}{\partial v^j} \frac{\partial f^i}{\partial v^k} = \frac{\partial f^k}{\partial v^j} \frac{\partial f^i}{\partial v^k} + \frac{\partial f^j}{\partial v^k} \frac{\partial f^i}{\partial v^j}$$

for $i\ j\ k$ distinct, that is

$$(2.29) \qquad \frac{\partial^2}{\partial v^j \partial v^k} e^{f^i} = \frac{\partial f^k}{\partial v^j} \frac{\partial e^{f^i}}{\partial v^k} + \frac{\partial f^j}{\partial v^k} \frac{\partial e^{f^i}}{\partial v^j} .$$

Define a new differential operator on any suitably differentiable function ψ by means of

$$(2.30) \qquad \underline{\psi_{i_1,\dots,i_n}} = e^{-\psi} \frac{\partial^n e^{\psi}}{\partial v^{i_1},\dots,\partial v^{i_n}} .$$

Note that although $\Psi_{\underline{i_1,\dots,i_{n+1}}} \neq (\Psi_{\underline{i_1,\dots,i_n}})_{\underline{i_{n+1}}}$

we have

$$(2.31) \quad \Psi_{\underline{i_1,\dots,i_{n+1}}} = (\Psi_{\underline{i_1,\dots,i_n}})_{i_{n+1}} + \Psi_{i_{n+1}} \Psi_{\underline{i_1,\dots,i_n}}$$

where the customary partial derivative is indicated by a subscript without underlining. In particular $\Psi_{\underline{j}} = \Psi_j$, $\Psi_{\underline{jk}} = \Psi_{jk} + \Psi_j \Psi_k$, and (2.28) becomes

$$(2.32) \qquad f^i_{\underline{jk}} = f^j_k f^i_{\underline{j}} + f^k_j f^i_{\underline{k}} \,.$$

Lemma C_1: Suppose Ψ satisfies

$$(2.33) \quad \Psi_{\underline{jk}} = f^j_k \Psi_{\underline{j}} + f^k_j \Psi_{\underline{k}}$$

and let $i_1,\dots,i_n$ be any distinct indices among $1,\dots,p$, say $1,\dots,n$ for convenience, $n \leq p$. Then

$$(2.34) \quad \Psi_{\underline{1,\dots,n}} = \sum_{i=1}^{n} \Psi_i \, f^i_{\underline{1,\dots,\hat{i},\dots,n}}$$

where $1,\dots,\hat{i},\dots,n$ indicates the $n-1$ integers omitting i. In particular

$$(2.35) \quad f^{n+1}_{\underline{1,\dots,n}} = \sum_{i=1}^{n} f^{n+1}_i \, f^i_{\underline{1,\dots,\hat{i},\dots,n}}$$

Proof: Use induction on n. By (2.33) the lemma holds for $n = 2$. To show that n may be replaced by $n+1$ in (2.34) use (2.31) to obtain

$$\psi_{\underline{1,\ldots,n+1}} = (\psi_{\underline{1,\ldots,n}})_{n+1} + \psi_{n+1}\,\psi_{\underline{1,\ldots,n}}$$

(2.36 $$= \sum_{i=1}^{n} (\psi_i F^i_{n+1} + \psi_{i,n+1} F^i + \psi_{n+1}\psi_i F^i)$$

$$= \sum_{i=1}^{n} \psi_i F^i_{n+1} + \sum_{i=1}^{n} \psi_{\underline{i,n+1}} F^i$$

where $F^i = f^i_{\underline{1,\ldots,\hat{i},\ldots,n}}$ for convenience.

But the second term on the right is

$$\sum_{i=1}^{n} (\psi_i f^i_{n+1} + \psi_{n+1} f^{n+1}_i)\, F^i$$

(2.37) $$= \sum_{i=1}^{n} \psi_i f^i_{n+1} F^i + \psi_{n+1} f^{n+1}_{\underline{1,\ldots,n}}$$

by (2.33) and (2.35), which is just the induction hypothesis applied to f^{n+1}. Hence by (2.31) and the induction hypothesis

$$\psi_{\underline{1,\ldots,n+1}} = \sum_{i=1}^{n} \psi_i (F^i_{n+1} + f^i_{n+1} F^i) + \psi_{n+1} f^{n+1}_{\underline{1,\ldots,n}}$$

(2.38) $$= \sum_{i=1}^{n} \psi_i f^i_{\underline{1,\ldots,\hat{i},\ldots,n+1}} + \psi_{n+1} f^{n+1}_{\underline{1,\ldots,n}}$$

$$= \sum_{i=1}^{n+1} \psi_i f^i_{\underline{1,\ldots,\hat{i},\ldots,n+1}}$$

as asserted.

Formula (2.35) may be iterated to find an expression for $f^{n+1}_{\underline{1,\ldots,n}}$ directly in terms of the f^j_i if desired.

If ψ satisfies (2.33), that is

$$(2.39)\qquad (e^{\psi})_{jk}\, e^{-\psi} = f^j_k\,(e^{\psi})_j\, e^{-\psi} + f^k_j\,(e^{\psi})_k\, e^{-\psi}$$

then clearly $\phi = e^{\psi}$ satisfies (2.27).

Similarly (2.34) becomes

$$(2.40)\qquad \phi_{1,\ldots,n} = \sum_{i=1}^{n} \phi_i\, f^i_{\underline{1,\ldots,\hat{i},\ldots,n}}$$

Since the right-hand side of (2.40) is symmetric in the indices $1,\ldots,n$ it is clear that the computation of $\phi_{1,\ldots,n}$ is independent of the order of differentiation. Thus any mixed derivative of ϕ involving no repeated differentiation may be uniquely evaluated in terms of the first derivatives of ϕ.

<u>THEOREM C</u>: Under the hypotheses of THEOREM B, if the integrability conditions (2.23) are satisfied, then a conservation law in a neighborhood of $u = u_0$ may be specified by assigning values of ϕ $(=\phi^2)$ on the p characteristic curves (2.16) through u_0. In particular, by successively determining conservation laws for which the derivatives of ϕ vanish along all but the k^{th} characteristic curve, $k = 1,\ldots,p$, it is clear that any system satisfying (2.39) is equivalent to infinitely many <u>systems</u> of conservation laws.

Proof: Write ϕ for $\phi(0,\ldots,0)$, ϕ^{ℓ} for $\phi(0,\ldots,\overset{\ell}{v},\ldots,0)$, $\phi^{\ell k}$ for $\phi(0,\ldots,\overset{\ell}{v},\ldots,\overset{k}{v},\ldots,0)$ etc., and ϕ_{ℓ} for $\dfrac{\partial\phi^{\ell}}{\partial v^{\ell}}$, $\phi_{\ell k}$ for $\dfrac{\partial^2\phi^{\ell k}}{\partial v^{\ell}\partial v^{k}}$, etc., and note the following sequence of integral formulae, which may be verified simply by evaluating the integrals:

$$\phi^{\ell} = \phi + \int_0^{v^{\ell}} \phi_{\ell}\, dv^{\ell} \tag{2.41}$$

$$\phi^{\ell k} = (\phi^{\ell} + \phi^{k}) - \phi + \int_0^{v^{\ell}}\int_0^{v^k} \phi_{\ell k}\, dv^{k} dv^{\ell} \tag{2.42}$$

$$\begin{aligned}\phi^{\ell kj} = {} & (\phi^{kj} + \phi^{j\ell} + \phi^{\ell k}) - (\phi^{\ell} + \phi^{k} + \phi^{j}) \\ & + \phi + \int_0^{v^{\ell}}\int_0^{v^k}\int_0^{v^j} \phi_{\ell kj}\, dv^{j} dv^{k} dv^{\ell},\end{aligned} \tag{2.43}$$

and so forth, until $\phi^{1,2,\ldots,p} = \phi(v^1,\ldots,v^p)$ at any point is given by

$$\begin{aligned}\phi^{1,\ldots,p} = {} & \sum(\text{all } \phi\text{'s with } p-1 \text{ entries}) \\ & - \sum(\text{all } \phi\text{'s with } p-2 \text{ entries}) + \cdots \\ & \cdots + (-1)^{p-1}\phi + \int_0^{v^1}\cdots\int_0^{v^p} \phi_{1,\ldots,p}\, dv^{p}\ldots dv^{1}.\end{aligned} \tag{2.44}$$

The data ϕ^{ℓ}, $\ell = 1,\ldots,p$ is given, so that (2.41) is trivial for all ℓ. Everything on the right-hand side of (2.42) except the integrand is therefore known. But according to the lemma the integrand is just a known linear combination of the first derivatives of ϕ which may be integrated by parts to give a linear integral equation in ϕ for which several existence and uniqueness proofs can be constructed. Thus all the $\phi^{\ell k}$ can be found, so that everything on the right-hand side of (2.43) except the integrand is known. Again the lemma allows the integrand to be uniquely expressed in terms of the first derivatives of ϕ which gives rise to a new integral equation, and so forth. This process successively finds the values of $\phi\ (v^1,\ldots,v^p)$ on higher and higher dimensional webs spanned by the characteristics, finally giving the values on the p-dimensional web, that is, in the entire space, at least in a neighborhood of $v = v_o$, depending on the method used to solve the integral equations.

CHAPTER III

THE GENERAL CASE

§1. Exterior differential forms

In Chapter I the existence of conservation laws was entirely reduced to the study of the linear system (1.23). This system is repeated here for convenience as

$$(3.1) \qquad \xi^{\ell}_{1} U_{\ell} \phi^1 + \xi^{\ell}_{2} U_{\ell} \phi^2 = 0, \qquad \ell = 1,\ldots,p,$$

where ξ^{ℓ}_{i} are functions of U, and the U_{ℓ} are linear differential first order operators, differentiations in the characteristic directions in U,

$$(3.2) \qquad U_{\ell} \equiv \eta^{\sigma}_{\ell} \frac{\partial}{\partial u^{\sigma}}, \qquad \ell = 1,\ldots,p,$$

spanning the tangent space of U, which is defined in Chapter I, §2.

The dual to the tangent space may be identified with the space of first order differential forms, called Pfaffian forms, over U, since the former transforms contravariantly and the latter covariantly under a change of coordinates, giving rise to appropriate bilinear functionals. It will be more convenient in the present chapter to work with the space of differential forms, spanned by the dual basis

$$(3.3) \qquad \omega^{\ell} \equiv \zeta^{\ell}_{\sigma} du^{\sigma}$$

where (ζ^{ℓ}_{σ}) is the inverse of (η^{σ}_{ℓ}), $\zeta^{\ell}_{\sigma}\eta^{\sigma}_{j} = \delta^{\ell}_{j}$.

This space is called the co-tangent space of U.

An *exterior differential form* of degree r in the variables $z^1,\dots,z^N$ is any element of the Grassman algebra of degree r generated by the space of Pfaffian forms in the N variables $z^1,\dots,z^N$. Thus, if the Pfaffian forms are spanned by $\theta_1,\dots,\theta_N$, an exterior differential form of degree r is a linear combination with analytic coefficients of terms $\theta_{i_1}\wedge\theta_{i_2}\wedge\cdots\wedge\theta_{i_r}$, where $i_1,\dots,i_r$ are integers from $1,\dots,N$, the sign "$\wedge$" denoting exterior product. The definition of exterior product assumes that

$$(3.4)\qquad \theta_{i_1}\wedge\theta_{i_2} + \theta_{i_2}\wedge\theta_{i_1} = 0.$$

These forms constitute a ring in which products are formed by exterior multiplication. Since (3.4) implies that the square of any element vanishes, it follows that this ring is of dimension $\binom{N}{1}+\binom{N}{2}+\cdots+\binom{N}{N} = 2^N-1$ considered as a vector space over the analytic functions, thus of dimension 2^N when the unit element is added, introducing the analytic functions as *differential forms of degree zero*.

From (3.4) one easily shows that the forms $\theta_1,\dots,\theta_r$ are linearly independent if and only if $\theta_1\wedge\theta_2\wedge\cdots\wedge\theta_r$ is a non-zero form.

The operation of *exterior differentiation*, denoted by d, is defined on a *monomial* $a\,dz^{i_1}\wedge\cdots\wedge dz^{i_r}$ by

$$(3.5)\qquad d\,(a dz^{i_1}\wedge\cdots\wedge dz^{i_r}) = \sum_{j=1}^{N}\frac{\partial a}{\partial z^j}\,dz^j\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_r}$$

and on arbitrary exterior differential forms by means of

$d(\Theta + \Theta') = d\Theta + d\Theta'$. Clearly

$$dd(a\,dz^{i_1}\wedge\cdots\wedge dz^{i_r}) = \sum_{j,k=1}^{N} \frac{\partial^2 a}{\partial z^k \partial z^j}\, dz^k\wedge dz^j\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_r}$$

$$(3.6) \qquad = 0$$

by (3.4), which gives Poincaré's theorem,

$$(3.7) \qquad dd\Theta = 0 ,$$

for an arbitrary exterior differential form Θ. If Θ is an exterior differential form of degree r and Θ' is an arbitrary exterior differential form, then the definition of exterior differentiation implies

$$(3.8) \qquad d(\Theta\wedge\Theta') = d\Theta\wedge\Theta' + (-1)^r\Theta\wedge d\Theta' .$$

A differential form Ω which is the exterior derivative of another differential form, $\Omega = d\Theta$, is called an integral, or total differential. Poincaré's theorem has a converse, namely, if $d\Omega = 0$ in a neighborhood of a given point, then Ω is an integral there, provided that the coefficients of Ω are analytic in the neighborhood.

The exterior derivative of any Pfaffian form is an exterior differential form of degree two. Consider the forms ω^l, $l = 1,\ldots,p$ given in (3.3) for example. We have

$$(3.9) \qquad d\omega^l = -\tfrac{1}{2}\, c^l_{\rho\sigma}\,\omega^\rho\wedge\omega^\sigma,$$

where we might as well take $c^{\ell}_{\rho\sigma} + c^{\ell}_{\sigma\rho} = 0$ due to (3.4). If f is an arbitrary analytic function of $u^1, \ldots, u^p$ then

$$(3.10)\qquad df = \frac{\partial f}{\partial u^{\sigma}}\, du^{\sigma} = \eta^{\sigma}_{\rho} \frac{\partial f}{\partial u^{\sigma}} \zeta^{\rho}_{\tau} du^{\tau} = U_{\sigma} f\, \omega^{\sigma}$$

so that

$$\begin{aligned}
0 = ddf &= d(U_{\sigma} f)_{\wedge} \omega^{\sigma} + U_{\sigma} f\, d\omega^{\sigma} \\
&= U_{\rho}\, (U_{\sigma} f) \omega^{\rho}{}_{\wedge} \omega^{\sigma} + U_{\tau} f\, d\omega^{\tau} \\
(3.11)\qquad &= \frac{1}{2} \left\{ U_{\rho}\, (U_{\sigma} f) - U_{\sigma} (U_{\rho} f) \right\} \omega^{\rho}{}_{\wedge} \omega^{\sigma} + U_{\tau} f\, d\omega^{\tau} \\
&= \frac{1}{2}\, [U_{\rho}\, U_{\sigma}] f\, \omega^{\rho}{}_{\wedge} \omega^{\sigma} + U_{\tau} f\, d\omega^{\tau} \\
&= \frac{1}{2} \gamma^{\tau}_{\rho\sigma}\, U_{\tau}\, f \omega^{\rho}{}_{\wedge} \omega^{\sigma} - \frac{1}{2} c^{\tau}_{\rho\sigma}\, U_{\tau} f \omega^{\rho}{}_{\wedge} \omega^{\sigma} \\
&= \frac{1}{2}\, (\gamma^{\tau}_{\rho\sigma} - c^{\tau}_{\rho\sigma})\, U_{\tau}\, f \omega^{\rho}{}_{\wedge} \omega^{\sigma}
\end{aligned}$$

by the definitions (1.29) and (3.9) of the structure functions $\gamma^{\tau}_{\rho\sigma}$ and the functions $c^{\tau}_{\rho\sigma}$ respectively. By definition $\gamma^{\tau}_{\rho\sigma} = c^{\tau}_{\rho\sigma} = 0$ for $\rho = \sigma$; and since (3.11) implies that the coefficient of $\omega^{\rho}{}_{\wedge}\omega^{\sigma}$ must vanish for $\sigma \neq p$ it follows that the $c^{\tau}_{\rho\sigma}$ of (3.9) are merely the structure functions,

$$(3.12)\qquad c^{\tau}_{\rho\sigma} = \gamma^{\tau}_{\rho\sigma}\,, \qquad\qquad \rho, \sigma, \tau = 1, \ldots, p.$$

§ 2. Frobenius' theorem

In Chapter II, §1, it was noted that the complete system (2.2), repeated here as

$$(3.13) \qquad W_\ell \Phi = 0, \qquad \ell = 1,\dots,r$$

for convenience, where $W_\ell = \sum_{j=1}^{N} \tau_\ell^j \frac{\partial}{\partial w^j}$ is a linear homogeneous first order differential operator in a neighborhood of w_0 in a manifold W of dimension N, possesses $N-r$ functionally independent solutions. It was further noted that there is a change of variables, $w = w(v)$, such that the given system becomes Jacobi complete,

$$(3.14) \qquad \frac{\partial \Phi}{\partial v^\ell} = 0, \qquad \ell = 1,\dots,r$$

The solutions Φ of this system are arbitrary functions of v^ℓ, $\ell = r+1,\dots,N$. Thus one can map a neighborhood of w_0, given by the coordinates $v_0 = (v_0^1,\dots,v_0^N)$, onto a neighborhood of some point $m_0 = (m_0^{r+1},\dots,m_0^N)$ in an $(N-r)$-dimensional manifold M by means of

$$(3.15) \qquad m^\ell = v^\ell, \qquad \ell = r+1,\dots,N.$$

M is the <u>integral manifold</u> determined by (3.13) in a neighborhood of w_0.

The fact that the vectors $\frac{\partial}{\partial v^\ell}$, $\ell = r+1,\dots,N$ span the tangent space of M in a neighborhood of w_0 is of no interest since these vectors depend on an a priori knowledge of M. However, the differentials $dv^\ell = dm^\ell$, $\ell = r+1,\dots,N$ span the corresponding co-tangent space and can be invariantly described as isomorphic to the annihilator of the vectors $\frac{\partial}{\partial v^{\ell}}$, $\ell = 1,\ldots,r$, given in (3.13) and (3.14), which determine subspaces of the tangent spaces in a neighborhood of w_o. Furthermore, since these differentials are independent their exterior product is non-zero, and so one could equally well describe M as the integral manifold through w_o for which $dv^{r+1} \wedge \ldots \wedge dv^N \neq 0$.

It is desirable to find a criterion for the completeness of (3.13) in terms of the annihilator **A** of $W_1,\ldots,W_r$. Suppose (3.13) is complete, and let $\theta_{r+1},\ldots,\theta_N$ be any basis of **A**. Then each θ_ℓ is some linear combination of $dv^{r+1},\ldots,dv^N$, where $v^{r+1},\ldots,v^N$ are any independent solutions of (3.13), with coefficients which are not a priori known. Since the square of any Pfaffian form vanishes, only one term appears in the product $\Omega = \theta_{r+1} \wedge \cdots \wedge \theta_N$, namely,

$$\Omega = \theta_{r+1} \wedge \cdots \wedge \theta_N = \lambda\, dv^{r+1} \wedge \cdots \wedge dv^N. \tag{3.16}$$

Thus **A** is completely described by a non-zero form Ω for some unknown factor λ, $\lambda \neq 0$. Differentiating the right-hand side of (3.16) according to (3.6), all terms except $d\lambda \wedge dv^{r+1} \wedge \cdots \wedge dv^N$ vanish by Poincaré's theorem. Hence

$$d\Omega = \frac{d\lambda}{\lambda} \wedge \Omega \tag{3.17}$$

so that $d\Omega$ is in the <u>ideal</u> generated by Ω over the ring of differential forms, which is also expressed by saying that the differential form Ω is <u>closed</u>. Conversely, if Ω is closed, <u>Frobenius' theorem</u> asserts that there is an integral manifold through w_o for which $\Omega \neq 0$, that is, satisfying (3.13).

As an application we return to the problem of Chapter II, §1 to determine when multiplication by an integrating factor and a change of variables $u = u(v)$ will bring the operators $U_\ell \equiv \eta_\ell^\sigma \frac{\partial}{\partial u^\sigma}$ into the form $\Lambda_\ell U_\ell \equiv \frac{\partial}{\partial v^\ell}$. If

$$U_\ell \equiv \frac{1}{\Lambda_\ell} \frac{\partial}{\partial v^\ell}, \qquad \ell = 1,\ldots,p \tag{3.18}$$

then the dual basis is of the form

$$\omega^\ell \equiv \zeta^\ell_\sigma \, du^\sigma \equiv \Lambda_\ell dv^\ell, \qquad \ell = 1,\ldots,p \tag{3.19}$$

Thus for each ℓ the differential $\Omega = \omega^\ell$ must be closed. That is, the exterior form

$$d\omega^\ell = -\frac{1}{2}\gamma^\ell_{\sigma\rho}\omega^\sigma \wedge \omega^\rho \tag{3.20}$$

must lie in the ideal generated by ω^ℓ, so that $\gamma^\ell_{\sigma\rho}$ must vanish except when $\sigma = \ell$ or $\rho = \ell$. This condition, found previously in Chapter II, is both necessary and sufficient by Frobenius' theorem.

§3. Exterior differential systems

Instead of prescribing only a form Ω which does not vanish on some unknown integral manifold one might also seek integral manifolds on which, in addition, a prescribed system of exterior differential forms does vanish. Problems of this type arise in a natural way from systems of linear homogeneous first order partial differential equations. For example, it is of interest to consider the system of Chapter I, §5, concerning the case $p = 2$,

$$(3.21)\qquad \xi^{\ell}_{1}\frac{\partial \phi^1}{\partial v^{\ell}} + \xi^{\ell}_{2}\frac{\partial \phi^2}{\partial v^{\ell}} = 0, \qquad \ell = 1,2$$

where ξ^{ℓ}_{i} are known functions of v such that $\begin{vmatrix} \xi^1_1 & \xi^1_2 \\ \xi^2_1 & \xi^2_2 \end{vmatrix} \neq 0.$

For simplicity we introduce the unsymmetry of Chapter II and take this in the form

$$(3.22)\qquad \frac{\partial \phi^1}{\partial v^{\ell}} + \xi^{\ell}\frac{\partial \phi^2}{\partial v^{\ell}} = 0, \qquad \ell = 1,2.$$

Let

$$(3.23)\qquad d\phi^2 = \phi_1\, dv^1 + \phi_2\, dv^2$$

then (3.22) implies

$$(3.24)\qquad d\phi^1 = -\xi^1\phi_1 dv^1 - \xi^2\phi_2 dv^2.$$

Suppose

$$(3.25)\qquad \begin{cases} d\phi_1 = \phi_{11}\, dv^1 + \phi_{12}\, dv^2 \\ d\phi_2 = \phi_{21}\, dv^1 + \phi_{22}\, dv^2\ , \end{cases}$$

then exterior differentiation of (3.23) gives

$$\begin{aligned} 0 &= (\phi_{11}dv^1 + \phi_{12}dv^2)\wedge dv^1 + (\phi_{21}dv^1 + \phi_{22}dv^2)\wedge dv^2 \\ (3.26)\qquad &= \phi_{12}dv^2\wedge dv^1 + \phi_{21}dv^1\wedge dv^2 \\ &= (\phi_{21} - \phi_{12})\, dv^1\wedge dv^2 \end{aligned}$$

so that

$$(3.27) \qquad \phi_{21} = \phi_{12}.$$

This was to be expected since it merely states that $\dfrac{\partial^2\phi}{\partial v^1 \partial v^2} = \dfrac{\partial^2\phi}{\partial v^2 \partial v^1}$.

Similarly exterior differentiation of (3.24) gives

$$(3.28) \qquad \xi^2_1\phi_2 + \xi^2\phi_{21} = \xi^1_2\phi_1 + \xi^1\phi_{12}$$

where $\quad d\xi^i = \xi^i_1 dv^1 + \xi^i_2 dv^2, \quad i = 1,2$

Hence, since by assumption $\xi^2 \neq \xi^1$,

$$(3.29) \qquad \phi_{21} = \phi_{12} = \mu^1\phi_1 + \mu^2\phi_2$$

where $\mu^1 = \dfrac{\xi^1_2}{\xi^2 - \xi^1}$ and $\mu^2 = \dfrac{\xi^2_1}{\xi^1 - \xi^2}$. Now substitute

these values into (3.25) and take exterior derivatives to find

$$(3.30) \qquad \begin{cases} 0 = d\phi_{11} \wedge dv^1 + \rho_1 dv^1 \wedge dv^2 \\ \\ 0 = \rho_2 dv^2 \wedge dv^1 + d\phi_{22} \wedge dv^2 \end{cases}$$

where

$$(3.31) \qquad \begin{cases} \rho_1 = \mu^1\phi_{11} + (\mu^1_1 + \mu^2\mu^1)\phi_1 + (\mu^2_1 + \mu^2\mu^2)\phi_2 \\ \\ \rho_2 = \mu^2\phi_{22} + (\mu^1_2 + \mu^1\mu^1)\phi_1 + (\mu^2_2 + \mu^1\mu^2)\phi_2. \end{cases}$$

Finally (3.23) and (3.30) may be taken together as a differential system in the four unknowns $\phi_1, \phi_2, \phi_{11}, \phi_{22}$, and two independent variables v^1 and v^2,

$$
(3.32)\qquad \begin{cases}
d\phi_1 - \phi_{11}dv^1 - (\mu^1\phi_1 + \mu^2\phi_2)dv^2 = 0 \\[2ex]
d\phi_2 - (\mu^1\phi_1 + \mu^2\phi_2)dv^1 - \phi_{22}dv^2 = 0 \\[2ex]
d\phi_{11}\wedge dv^1 + \rho_1 dv^1\wedge dv^2 = 0 \\[2ex]
\rho_2 dv^2\wedge dv^1 + d\phi_{22}\wedge dv^2 = 0 .
\end{cases}
$$

The relations $d(\phi_1 dv^1 + \phi_2 dv^2) = 0$ and $d(\xi^1\phi_1 dv^1 + \xi^2\phi_2 dv^2) = 0$ imply that the two Pfaffian forms in parentheses are integrals, by the converse of Poincaré's theorem. Since these two equations can be obtained without differentiation from (3.32), we have omitted (3.23) and (3.24) from the collection (3.32). In fact this system already describes the original unknowns ϕ^1 and ϕ^2 up to additive constants of no interest in conservation laws.

Now note that further exterior differentiation of (3.32) yields only equations which are already in the ideal it generates over the ring of exterior forms in the space $(v^1, v^2, \phi_1, \phi_2, \phi_{11}, \phi_{22})$, so that the system (3.32) is closed. If an integral manifold can be found which satisfies (3.32) and on which in addition $dv^1\wedge dv^2 \neq 0$, then the system is said to be in involution with respect to the variables v^1 and v^2. Clearly if (3.32) is in involution with respect to v^1 and v^2 then there exist solutions of (3.21). The number of such solutions as well as a criterion for a closed system to be in involution is given in §4.

An important detail in the preceding derivation should be stressed. The original differential system, comprising (3.23) and (3.24), was not closed, but it might have been closed merely by adding the exterior derivatives of these equations to it, giving the system

$$
(3.33) \quad \begin{cases}
d\phi^2 - \phi_1 dv^1 - \phi_2 dv^2 = 0 \\
d\phi^1 + \xi^1 \phi_1 dv^1 + \xi^2 \phi_2 dv^2 = 0 \\
d\phi_1 \wedge dv^1 + d\phi_2 \wedge dv^2 = 0 \\
\xi^1 d\phi_1 \wedge dv^1 + \xi^2 d\phi_2 \wedge dv^2 + (\phi_2 \xi^2_1 - \phi_1 \xi^1_2) dv^1 \wedge dv^2 = 0,
\end{cases}
$$

in which the first two equations might be omitted as before. As it turns out, (3.33) is not in involution, so that it has been necessary to <u>prolong</u> it by adding ϕ_{11} and ϕ_{22} to it by means of (3.25). Then the new system had to be closed by the addition of the exterior derivatives of (3.25); luckily the resulting system is in involution as we shall see later on. It is an open question whether an arbitrary system can be prolonged in this fashion into a system in involution with respect to a given set of variables.

The remainder of this section is devoted to replacing the general equations (3.1) governing the existence of conservation laws by a closed exterior differential system. Since one may solve (3.1) for $U_\ell \phi^i$ in terms of some new parameters, χ_ℓ say,

$$U_\ell \phi^1 = \chi_\ell \xi^\ell_2$$
(3.34)
$$U_\ell \phi^2 = -\chi_\ell \xi^\ell_1 ,$$

the initial differential system in the variables $(\phi^1, \phi^2, \chi_1, \ldots, \chi_p, u^1, \ldots, u^p)$ is

$$d\phi^1 - \sum_{\ell=1}^{p} \chi_\ell \xi^\ell_2 \omega^\ell = 0$$
(3.35)
$$d\phi^2 + \sum_{\ell=1}^{p} \chi_\ell \xi^\ell_1 \omega^\ell = 0$$

where $\omega^\ell = \sum_{j=1}^{p} \zeta^\ell_j du^j$. The system (3.35) can also be obtained directly from (1.9) without going to the trouble of deriving the corresponding linear equations (3.1). In fact, write (1.9) in the normal form

(1.18) $\quad \xi^\ell_2 \sigma^\ell_1 - \xi^\ell_1 \sigma^\ell_2 = 0, \qquad \ell = 1, \ldots, p$

where $\sigma^\ell_i = \sum_{j=1}^{p} \zeta^\ell_j \frac{\partial u^j}{\partial x^i}$, and suppose there exists a linear combination $\sum_{\ell=1}^{p} \chi_\ell (\xi^\ell_2 \sigma^\ell_1 - \xi^\ell_1 \sigma^\ell_2)$ of the left-hand members of (1.18) which is of the form $\frac{\partial \phi^1}{\partial x^1} + \frac{\partial \phi^2}{\partial x^2}$. Then

$$\frac{\partial \phi^1}{\partial x^1} \equiv \sum_{\ell=1}^{p} \chi_\ell \xi^\ell_2 \sigma^\ell_1 \qquad \text{and} \qquad \frac{\partial \phi^2}{\partial x^2} \equiv - \sum_{\ell=1}^{p} \chi_\ell \xi^\ell_1 \sigma^\ell_2 ;$$

since each of these identities involves differentiation with respect to only one of the independent variables, we may replace the derivatives $\frac{\partial \phi^i}{\partial x^i}$ and σ^{ℓ}_{i} by the corresponding differentials $d\phi^i$ and ω^{ℓ} to obtain (3.35).

We attempt to prolong (3.35) by adding new variables $\chi_{\ell\rho}$ to it, where

$$(3.36) \qquad d\chi_{\ell} = \chi_{\ell\rho}\,\omega^{\rho}, \qquad \ell = 1,\ldots,p,$$

Denote the Pfaffian form on the right-hand side of (3.36) by θ_{ℓ}. The introduction of θ_{ℓ} is convenient for deriving results which are independent of the fact that it is an integral; it should be noted in particular that Poincaré's theorem, $dd\,\theta_{\ell} = 0$, is always valid, even if nothing is known about θ_{ℓ}.

Differentiate (3.35) to obtain

$$(3.37) \qquad \sum_{\ell=1}^{p} \left\{ \theta_{\ell} \wedge \xi^{\ell}\,\omega^{\ell}_{i} + \chi_{\ell}\, d\,(\xi^{\ell}_{i}\,\omega^{\ell}) \right\} = 0, \qquad i = 1,2.$$

The coefficient of $\omega^{\ell} \wedge \omega^{k}$ in (3.37) is given in terms of the χ_{ℓ} and the known functions $\gamma^{\ell}_{jk}, \xi^{\ell}_{i}, \xi^{\ell}_{ik}$ in the second term, and by $\chi_{k\ell}\,\xi^{k}_{i} - \chi_{\ell k}\,\xi^{\ell}_{i}$ in the first term, $i = 1,2$. Hence, since the characteristic directions in X are distinct, by setting the coefficients of $\omega^{\ell} \wedge \omega^{k}$ equal to zero, $k, \ell = 1,\ldots,p$, one can solve for all those $\chi_{k\ell}$ for which $k \neq \ell$ in terms of the χ_{ℓ}. Taking these values for $\chi_{k\ell}$, or, preferably, merely considering these values as definitions, (3.37) is identically satisfied and hence the exterior derivative

$$(3.38) \qquad \sum_{\ell=1}^{p} \left\{ d\theta_{\ell} \wedge \xi^{\ell}_{i}\,\omega^{\ell} - \theta_{\ell} \wedge d(\xi^{\ell}_{i}\,\omega^{\ell}) + d\chi_{\ell} \wedge d(\xi^{\ell}_{i}\,\omega^{\ell}) \right\} \equiv 0$$

of the left-hand side is identically zero. (Note that a minus sign appears in (3.38) due to (3.8).) In other words, since the last two terms of (3.38) cancel, using (3.36) and the definition of θ_ℓ, the expressions

$$(3.39) \qquad \sum_{\ell=1}^{p} d\theta_\ell \wedge \xi_i^\ell \omega^\ell$$

are in the ideal generated by

$$(3.40) \qquad d\chi_\ell - \theta_\ell, \qquad \ell = 1,\ldots,p.$$

For the converse of Poincaré's theorem turns (3.35) into (3.37), and (3.37) vanishes identically, merely being used to define various coefficients in the θ_ℓ's. Note that the system (3.40) is not yet closed, although θ_ℓ represents an integral, since θ_ℓ involves new terms which prolong the original system. In the prolonged system θ_ℓ is not an integral.

Now let

$$(3.41) \qquad d\theta_\ell = \tilde{\omega}_{\ell\ell} \wedge \omega^\ell + \frac{1}{2} \sum_{j,k \neq \ell} \pi_{\ell jk}\, \omega^j \wedge \omega^k$$

for some $\tilde{\omega}_{\ell\ell}$, $\pi_{\ell jk}$ depending on χ_ℓ, $\chi_{\ell\ell}$ and their first differentials, $\ell, j, k = 1,\ldots,p$, where $\pi_{\ell jk} + \pi_{\ell kj} = 0$. Without yet closing (3.40) by asserting that $d\theta_\ell = 0$, multiply (3.41) by ω^ℓ to find that

$$(3.42) \qquad \frac{1}{2} \sum_{\ell,j,k=1}^{p} \xi_i^\ell\, \pi_{\ell jk}\, \omega^j \wedge \omega^k \wedge \omega^\ell$$

lies in the ideal generated by (3.40), since (3.39) does;

hence

$$\xi_i^{\ell}\,\pi_{\ell jk} + \xi_i^{j}\,\pi_{jk\ell} + \xi_i^{k}\,\pi_{k\ell j} = 0, \qquad i = 1,2. \tag{3.43}$$

But (3.43) gives the identity

$$\frac{\pi_{\ell jk}}{\begin{vmatrix} \xi_1^j & \xi_1^k \\ \xi_2^j & \xi_2^k \end{vmatrix}} = \frac{\pi_{jk\ell}}{\begin{vmatrix} \xi_1^k & \xi_1^{\ell} \\ \xi_2^k & \xi_2^{\ell} \end{vmatrix}} = \frac{\pi_{k\ell j}}{\begin{vmatrix} \xi_1^{\ell} & \xi_1^j \\ \xi_2^{\ell} & \xi_2^j \end{vmatrix}} = \pi(\ell jk) \tag{3.44}$$

where $\pi(\ell jk)$ is independent of the order of ℓjk. In the special case of Chapter II this simple result is exactly the assertion of Chapter II, § 2, that only one first order relation arises for each distinct triple of indices. In fact, $\pi(ijk)$ is just the left-hand side of (2.37). It should be noted, however, that in general π is a linear combination of the $x_{\ell\ell}$ as well as x_{ℓ}, $\ell = 1,\ldots,p$, the special case being the only exception to this rule.

It is undoubtedly true that for $p > 3$ one can find still further identities among the $\pi(\ell jk)$. These identities, if any, will not be investigated here.

The fact that $dd\Theta_{\ell}$ belongs to the ideal generated by (3.40), that is

$$\begin{aligned} &d\,(\tilde{\omega}_{\ell\ell}\wedge\omega^{\ell}) + \frac{1}{2}\sum_{j,k\neq\ell} d\,\pi_{\ell jk}\,\omega^j\wedge\omega^k \\ &+ \frac{1}{2}\sum_{j,k,\neq\ell} \pi_{\ell jk}\,d\,(\omega^j\wedge\omega^k) = 0, \end{aligned} \tag{3.45}$$

will be recorded here for later use.

To close (3.40) one must first add to it the requirement $d\Theta_\ell = 0$, which breaks into several parts in the notation of (3.41). The result is that

$$(3.46)\qquad \begin{cases} \pi(\ell,j,k) = 0 \\ dx_\ell - \theta_\ell = 0 \\ \tilde{\omega}_{\ell\ell}\wedge\omega^\ell = 0 \end{cases} \qquad \ell,j,k = 1,\dots,p$$

is a closed system provided that $d(\tilde{\omega}_{\ell\ell}\wedge\omega^\ell)$ and $d\pi(\ell jk)$ belong to the ideal generated by it. However, it is not necessary to look at $d(\tilde{\omega}_{\ell\ell}\wedge\omega^\ell)$ since

$$d(\tilde{\omega}_{\ell\ell}\wedge\omega^\ell) + \frac{1}{2}\sum_{j,k\neq\ell} d\pi_{\ell jk}\,\omega^j\wedge\omega^k, \qquad \ell = 1,\dots,p$$

belongs to the ideal generated by the left-hand members of (3.46), thanks to (3.45). Hence $d(\tilde{\omega}_{\ell\ell}\wedge\omega^\ell)$ vanishes if $d\pi(\ell jk)$ does for all $j,k = 1,\dots,p$.

Finally, if $d\pi(\ell jk)$ vanishes then so does its derivative, by Poincaré's theorem, since no prolongation of the system is involved at this point. Thus

$$(3.47)\qquad \begin{cases} \pi(\ell,j,k) = 0 \\ d\pi(\ell,j,k) = 0 \\ dx_\ell - \theta_\ell = 0 \\ \tilde{\omega}_{\ell\ell}\wedge\omega^\ell = 0 \end{cases} \qquad \ell,j,k = 1,\dots,p$$

is a closed system in $\chi_\ell, \chi_{\ell\ell}$, and u^ℓ, $\ell = 1,\dots,p$, all of whose solutions are also solutions of (3.1). The main result of this chapter is that (3.47) is also always <u>in involution</u> with respect to the u^ℓ.

§4. The theorem of Cartan and Kähler

The following existence theorem, due to Cartan [1] and extended by Kähler [2] to arbitrary exterior differential systems, not only gives a criterion for a closed system to be in involution at a given point, but also indicates how much Cauchy data (in the form of coefficients of convergent power series in one or more variables about the point) is necessary to determine a unique solution of the system. The presentation given here for the case that the system contains no forms of degree higher than two is essentially that given in Kähler, pp. 54-55, with only slight changes of notation. Frobenius' theorem and the converse of Poincaré's theorem can be obtained as special cases of this theorem.

Given p independent Pfaffian forms $\omega^1,\dots,\omega^p$ in a certain neighborhood in some space of dimension $n + r$, we consider a system

$$
(3.48)\quad \begin{cases} \pi_i = 0, & i = 1,\dots,r \\ \theta_j = 0, & j = 1,\dots,h \\ \chi_k = 0, & k = 1,\dots,m \end{cases}
$$

of forms π_i, θ_j, χ_k of degree zero, one, and two respectively, where the Jacobian matrix formed by the gradients of the forms of degree zero is of rank r, the $p+h$ Pfaffian forms $\omega^1,\ldots,\omega^p$, $\theta_1,\ldots,\theta_h$ are linearly independent, the quadratic forms are of the type $\chi_k = \sum_{\ell=1}^{p} \tilde{\omega}_{k\ell} \wedge \omega^\ell$ for some new Pfaffian forms $\tilde{\omega}_{k\ell}$, and the coefficients of all of the forms are analytic. We seek a solution of (3.48) on which $\omega^1 \wedge \ldots \wedge \omega^p \neq 0$.

Let $q = n - p - h$ and suppose that new independent Pfaffian forms $\tilde{\omega}_1,\ldots,\tilde{\omega}_q$ are chosen in such a way that $\omega^1,\ldots,\omega^p$, $\theta_1,\ldots,\theta_h$, $\tilde{\omega}_1,\ldots,\tilde{\omega}_q$ span the co-tangent space of the n-dimensional manifold determined by the zero-degree forms

$$(3.49) \qquad \pi_i = 0, \qquad i = 1,\ldots,r.$$

Then the Pfaffian forms $\tilde{\omega}_{k\ell}$ are linear combinations of $\omega = (\omega^1,\ldots,\omega^p)$, $\theta = (\theta_1,\ldots,\theta_h)$ and $\tilde{\omega} = (\tilde{\omega}_1,\ldots,\tilde{\omega}_q)$; since only the coefficients of $\tilde{\omega}$ are of interest, and only the manifold determined by setting $\pi = (\pi_1,\ldots,\pi_r) = 0$, we may write

$$(3.50) \qquad \tilde{\omega}_{k\ell} \equiv \sum_{j=1}^{q} a^j_{k\ell}\, \tilde{\omega}_j \quad (\text{mod } \omega, \theta, \pi)$$

$$k = 1,\ldots,m; \qquad \ell = 1,\ldots,p.$$

Let σ_1 represent the number of linearly independent forms among

$\sum_{\ell=1}^{p} \tilde{\omega}_{k\ell}\, u_1^\ell$, mod ω, θ, π, $k = 1,\ldots,m$, where $u_1 = (u_1^1,\ldots,u_1^p)$ is a set of constants chosen in such a way that σ_1 is a maximum, or, preferably, u_1 is a fixed set of indeterminates. Similarly let $\sigma_1 + \sigma_2$ represent the number of linearly independent forms among $\sum_{\ell=1}^{p} \tilde{\omega}_{k\ell} u_1^{\ell}$ and $\sum_{\ell=1}^{p} \tilde{\omega}_{k\ell} u_2^{\ell}$ mod ω, θ, π, where u_2 is chosen in such a way that σ_2 is a maximum,..., let $\sigma_1 + \ldots + \sigma_{p-1}$ represent the number of linearly independent forms among $\sum_{\ell=1}^{p} \tilde{\omega}_{k\ell} u_1^{\ell}, \ldots,$ $\sum_{\ell=1}^{p} \tilde{\omega}_{k\ell} u_{p-1}^{\ell}$, mod ω, θ, π, where u_{p-1} is chosen in such a way that σ_{p-1} is a maximum. Finally let $\sigma_p = q - \sigma_1 - \ldots - \sigma_{p-1}$.

One further definition is necessary. An integral of (3.48) on which $\omega^1 \wedge \ldots \wedge \omega^p \neq 0$ is completely specified once $\tilde{\omega}$ is written in terms of ω, say,

$$(3.51) \qquad \tilde{\omega}_j = \sum_{\ell=1}^{p} t_{j\ell}\, \omega^{\ell}, \qquad j = 1, \ldots, q,$$

essentially prolonging (3.48) by adding the $t_{j\ell}$ to it. However, the $t_{j\ell}$ are restricted by the quadratic relations in (3.48) and so do not in general span a space of pq dimensions. Let M be the dimension of the space spanned by the $t_{j\ell}$.

The Cartan-Kähler theorem asserts that

$$M \leq pq - (p-1)\sigma_1 - (p-2)\sigma_2 - \dots - \sigma_{p-1}, \tag{3.52}$$

with equality holding if and only if (3.48) is in involution with respect to p variables for which $\omega^1 \wedge \dots \wedge \omega^p \neq 0$. In the latter case the general solution depends on σ_o arbitrary parameters, where $\sigma_o = n - p - q$, on σ_1 arbitrary analytic functions of a single variable, on σ_2 arbitrary analytic functions of two variables,..., and on σ_p arbitrary analytic functions of p variables.

This theorem is proved by means of a sequence of Cauchy-Kowalewski constructions. Setting

$$\omega^\ell = \sum_{k=1}^{p} u_k^\ell \omega_*^k$$ one successively constructs "integral elements"

through the spaces determined by $\omega_*^{k+1} = \dots = \omega_*^p = 0$ for $k = 1,\dots,p$.

It should be remarked that even though a closed system is in involution, further prolongation might still lead to a new system in involution. In particular, the singular solutions of a system in involution, which satisfy additional restrictions under which the given system is no longer in involution, can sometimes be obtained as the general solutions of a new prolonged system in involution.

§5. The existence of conservation laws

As a simple illustration the Cartan-Kähler theorem can now easily be applied to the first example of §3, given by

$$(3.32)\qquad \begin{cases} d\phi_1 - \phi_{11}dv^1 - (\mu^1\phi_1 + \mu^2\phi_2)dv^2 = 0 \\ d\phi_2 - (\mu^1\phi_1 + \mu^2\phi_2)dv^1 - \phi_{22}dv^2 = 0 \\ (d\phi_{11} - \rho_1 dv^2)_\wedge dv^1 = 0 \\ (d\phi_{22} - \rho_2 dv^1)_\wedge dv^2 = 0. \end{cases}$$

Since there are no forms of degree zero, $r = 0$. The variables occurring in (3.32) are $v^1, v^2, \phi_1, \phi_2, \phi_{11}, \phi_{22}$ so that $n = 6$. A solution is sought on which $dv^1{}_\wedge dv^2 \neq 0$, so $p = 2$. The system (3.32) contains two independent Pfaffian forms, giving $h = 2$ and $q = n - p - h = 2$. Clearly we may set $\tilde{\omega}_\ell = d\phi_{\ell\ell}$, with $\tilde{\omega}_{\ell\ell} \equiv \tilde{\omega}_\ell$ (mod ω, θ), $\ell = 1,2$; thus $\sigma_1 = 2$ and $\sigma_2 = q - \sigma_1 = 0$. Finally $d\phi_{11}$ and $d\phi_{22}$ are completely determined by $d\phi_{11} = t_{11}dv^1 + \rho_1 dv^2$ and $d\phi_{22} = \rho_2 dv^1 + t_{22}dv^2$, where t_{11} and t_{22} are unrestricted by the quadratic members of (3.32), giving $M = 2$. Thus equality holds in (3.52) and the general solution of (3.32) depends on two parameters, and two functions of a single variable, since $n - p - q = 2$ and $\sigma_1 = 2$.

More generally, the special case of Chapter II follows exactly the same pattern. Again there are no forms of degree zero, $\pi = 0$, provided the integrability conditions of THEOREM B are satisfied. Referring to (3.47), it is clear that each $\tilde{\omega}_{\ell\ell}$ contains only one term of interest, namely $d\chi_{\ell\ell}$, which was defined in (3.36), so that we may take $\tilde{\omega}_\ell = d\chi_{\ell\ell}$ with $\tilde{\omega}_{\ell\ell} \equiv \tilde{\omega}_\ell$ (mod ω, θ), $\ell = 1,\dots,p$. Then exactly as before, $n = 3p$, $h = p$, $q = n - p - h = p$, $\sigma_1 = p$, $\sigma_2 = \dots = \sigma_p = 0$, since $\sigma_1 + \dots + \sigma_p \leq q$, and $M = p$. The equality holds in (3.52) and so the general solution depends on p parameters and p functions of a single variable. This result might be slightly surprising since the existence theorem given in Chapter II indicates that the values of the first derivatives of χ along the p characteristic curves determine a solution uniquely up to two additive constants which do not occur in the present discussion. However, if one assigns not the first but the second derivatives $\chi_{\ell\ell}$ along the characteristic curves, the p values of the first derivatives at the origin may be taken as the additional parameters.

As a slightly more interesting example consider the special case again but suppose that none of the integrability conditions is satisfied. Then, since the coefficient of χ_ℓ occurring in $\pi(\ell jk)$ is non-zero, the equations $d\pi(\ell jk) = 0$ in (3.47) provide a determination of all the $\chi_{\ell\ell}$ in terms of the χ_ℓ, the determination being unique since (3.47) is closed. Because the system already contains expressions for $d\chi_\ell$, namely $d\chi_\ell = \theta_\ell$, the quadratic members of (3.47) are redundant and may be dropped out. When no quadratic terms are present both sides of (3.52) vanish so that the system is <u>automatically</u> in involution. No arbitrary functions may be assigned, since $q = 0$, and the number of arbitrary parameters depends on the equations $\pi(\ell jk) = 0$. The number of these equations is $\binom{p}{3}$, but there might be considerable linear dependence among them, as suggested on p. 53. However, for $p = 3$ there is exactly one relation among the three unknowns so that a two-parameter family of conservation laws exists.

To illustrate the preceding result consider the system

$$(3.53)\quad \begin{cases} \dfrac{\partial \phi^1}{\partial v^1} + (v^2 - v^3)\dfrac{\partial \phi^2}{\partial v^1} = 0 \\[2ex] \dfrac{\partial \phi^1}{\partial v^2} + v^1 \dfrac{\partial \phi^2}{\partial v^2} = 0 \\[2ex] \dfrac{\partial \phi^1}{\partial v^3} - v^1 \dfrac{\partial \phi^2}{\partial v^3} = 0 \end{cases}$$

which satisfies none of the conditions of THEOREM B. In fact it leads to the first order equation $\pi = 0$ where

$$(3.54)\quad \pi = 2(v^2 - v^3)\frac{\partial \phi^2}{\partial v^1} + (-v^1 - v^2 + v^3)\frac{\partial \phi^2}{\partial v^2} + (v^1 - v^2 + v^3)\frac{\partial \phi^2}{\partial v^3}.$$

Nevertheless it has two solutions,

$$(3.55)\quad \begin{cases} \phi^1 = v^1 (v^3 - v^2) \\ \phi^2 = v^1 + v^2 + v^3 \end{cases}$$

and

$$(3.56)\quad \begin{cases} \phi^1 = v^1 (v^3 - v^2)(v^1 + v^2 + v^3) \\ \phi^2 = (v^1)^2 + (v^2)^2 + (v^3)^2 + v^1 (v^2 + v^3), \end{cases}$$

the general solution being a linear combination of these with constant coefficients.

Intermediate cases, in which some but not all of the integrability conditions of THEOREM B are satisfied, can be investigated in exactly the same fashion. In every case, of course, the conservation laws fall short of equivalence to the original partial differential system unless all of the integrability conditions are satisfied.

Some examples not contained in the special case will be given in order to motivate the theorem to follow. In giving these examples it will be convenient to specify the differential forms $\omega^1,\dots,\omega^p$, or equivalently, the operators $U_1,\dots,U_p$, merely by giving their structure functions, which will be taken as constants satisfying the quadratic relations arising from Jacobi's identity, (1.32). The fundamental theorem of Lie groups insures the existence of $\omega^1,\dots,\omega^p$ in an appropriate neighborhood. We take $p = 3$ and suppose that the ratios $\xi^\ell_1 : \xi^\ell_2$ are distinct constants. Further we suppose for simplicity that the structure constants satisfy a condition which is just the opposite of that assumed in Chapter II, namely that γ^i_{jk} is <u>non</u>-zero only when i j k are all distinct. For convenience let (ijk) be any even permutation of (123) and write

$$\gamma^i_{jk} = \gamma^i \tag{3.57}$$

and

$$\bar{\chi}_i = \frac{1}{\begin{vmatrix} \xi^j_1 & \xi^k_1 \\ \xi^j_2 & \xi^k_2 \end{vmatrix}} \chi_i \, . \tag{3.58}$$

Then differentiating (3.35) gives

$$d\bar{\chi}_i = \bar{\theta}_i = \bar{\chi}_{ii}\omega^i + \begin{vmatrix} \omega^j & \omega^k \\ \gamma^j \bar{\chi}_j & \gamma^k \bar{\chi}_k \end{vmatrix} , \tag{3.59}$$

and further differentiation gives

$$(3.60) \qquad \pi = \gamma^i \bar{\chi}_{ii} + \gamma^j \bar{\chi}_{jj} + \gamma^k \bar{\chi}_{kk}$$

and

$$(3.61) \qquad \tilde{\omega}_{ii} \wedge \omega^i = 0, \qquad i = 1,2,3$$

where

$$(3.62) \qquad \tilde{\omega}_{ii} = d\bar{\chi}_{ii} - 2\gamma^j \gamma^k (\bar{\chi}_j \omega^j + \bar{\chi}_k \omega^k).$$

Now we consider various examples obtained by substituting these values of θ_i, $\tilde{\omega}_{ii}$, and π into the closed system

$$(3.47) \qquad \begin{cases} \pi = 0 \\ d\pi = 0 \\ d\bar{\chi}_i - \bar{\theta}_i = 0 \\ \tilde{\omega}_{ii} \wedge \omega^i = 0, \end{cases}$$

for several choices of $\gamma^1, \gamma^2, \gamma^3$. The special case $\gamma^1 = \gamma^2 = \gamma^3 = 0$ is omitted here.

Example I: Suppose $\gamma^1 = 1$, $\gamma^2 = \gamma^3 = 0$. Then $\tilde{\omega}_{11}$ is known in terms of the eight variables $u^1, u^2, u^3, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3, \bar{\chi}_{22}, \bar{\chi}_{33}$ since (3.60) implies $\bar{\chi}_{11} = 0$. Hence there are only two quadratic relations in (3.61). Furthermore $q = n - p - h = 8 - 3 - 3 = 2$ so that since σ_1 is clearly 2, $\sigma_2 = \sigma_3 = 0$. Since the components of $d\bar{\chi}_{22}$ and $d\bar{\chi}_{33}$ in the directions ω^2 and ω^3 respectively may be arbitrarily prescribed, $M = 2$. Hence $M = pq - (p-1)\sigma_1 = 2$ so that the system is in involution. Furthermore, three parameters and two analytic functions of a single variable determine the corresponding conservation laws since $n - p - q = 3$ and $\sigma_1 = 2$. It is interesting to note that $d\bar{\chi}_1 = 0$, so that $\bar{\chi}_1$ is one of the parameters. A more careful examination of the Cartan-Kähler theorem would probably show that the values of $\bar{\chi}_2$ and $\bar{\chi}_3$ at a fixed point are the remaining parameters, giving rise to an equivalent system of conservation laws.

Example II: Suppose $\gamma^1 = \gamma^2 = 1$, $\gamma^3 = 0$. Here we note that $\tilde{\omega}_{11} = d\bar{\chi}_{11}$ and $\tilde{\omega}_{22} = d\bar{\chi}_{22}$ so that $d\pi = 0$ implies $\tilde{\omega}_{11} + \tilde{\omega}_{22} = 0$. Since the quadratic relations show that $\tilde{\omega}_{11}$ and $\tilde{\omega}_{22}$ lie in different directions in the co-tangent space this means $\tilde{\omega}_{11} = \tilde{\omega}_{22} = 0$, leaving only one quadratic relation, $\tilde{\omega}_{33} \wedge \omega^3 = 0$. As before there are eight variables, which may be taken as $u^1, u^2, u^3, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3, \bar{\chi}_{22}, \bar{\chi}_{33}$ due to the relation $\pi = 0$. In a fashion similar to the first example one computes $M = q = \sigma_1 = 1$, where it should be noted that the additional Pfaffian relation $d\pi = 0$ gives $h = 4$ rather than $h = 3$. Thus $M = pq - (p-1)\sigma_1 = 1$, so that the system is in involution. One may prescribe three arbitrary parameters and one analytic functions of a single variable in order to determine the solution, since $n - p - q = 3$ and $\sigma_1 = 1$.

Example III: Suppose $\gamma^1 = \gamma^2 = \gamma^3 = 1$. In this case $\tilde{\omega}_{ii} = d\bar{\chi}_{ii} - 2(\bar{\chi}_j\omega^j + \bar{\chi}_k\omega^k)$, according to (3.62), for $i = 1,2,3$, so that the Pfaffian equation $d\pi = 0$ becomes $(\tilde{\omega}_{11} + \tilde{\omega}_{22} + \tilde{\omega}_{33}) + 4(\bar{\chi}_1\omega^1 + \bar{\chi}_2\omega^2 + \bar{\chi}_3\omega^3) = 0$. Since $\tilde{\omega}_{ii}$ is in the direction of ω^i this breaks into three Pfaffian relations, namely $\tilde{\omega}_{ii} + 4\bar{\chi}_i \omega^i = 0$, $i = 1,2,3$. Thus all the $\tilde{\omega}_{ii}$ are known, and the quadratic relations are merely redundant. Note that although six Pfaffian forms are involved in the system itself only five of them are independent since the relation $d\pi = 0$ is merely the derivative of the definition of χ_{11}, say, which is not a prolongation. Thus clearly $q = n - p - h = 8 - 3 - 5 = 0$ and $M = \sigma_1 = 0$ so that the system is in involution and is completely determined by five parameters since $n - p - q = 5$.

In every example so far the relations $m = M = \sigma_1 = q$ and $\sigma_2 = \ldots = \sigma_p = 0$ have been satisfied, where m is the number of independent quadratic relations, so that the condition $M = pq - (p-1)\sigma_1 - \ldots - \sigma_{p-1}$ has always been satisfied by virtue of $m = pm - (p-1)m$. It is clear that $\sigma_1 = m$ always holds since each independent $\tilde{\omega}_{\ell\ell}$ contains exactly one term $d\chi_{\ell\ell}$ independent of the Pfaffian forms $\omega^1, \ldots, \omega^p, \theta_1, \ldots, \theta_h$ and the other $\tilde{\omega}_{\ell\ell}$. For the same reason $\sigma_2 = \ldots = \sigma_{p-1} = 0$. Furthermore $M = m$ always holds since the component of $d\chi_{\ell\ell}$ in every direction except ω^ℓ is determined for each term occurring in the m independent quadratic relations; that is, in the language of §4, there are m independent parameters $t_{\ell\ell}$. Finally, to show that $q = m$ in every case, suppose the opposite, that is, $q > m$ since $q < m$ is impossible by the Cartan-Kähler inequality (3.52). In this case, since $q = \sigma_1 + \ldots + \sigma_p$ and $\sigma_1 + \ldots + \sigma_{p-1} = m$ one could prescribe σ_p arbitrary analytic functions of p variables in determining the conservation laws, $\sigma_p > 0$. This means that ϕ^2 for example, could be completely specified in a neighborhood of U, which is clearly impossible because ϕ^2 satisfies several second order equations. Thus the following important result is clear.

<u>THEOREM D</u>: The system

$$(3.47)\qquad \begin{cases} \pi(\ell,j,k) = 0 \\ d\pi(\ell,j,k) = 0 \\ d\chi_\ell - \theta_\ell = 0 \\ \tilde{\omega}_{\ell\ell} \wedge \omega^\ell = 0 \end{cases} \qquad \ell,j,k = 1,\ldots,p$$

is always in involution.

The importance of this result is that it gives a perfectly definite criterion for establishing the existence of conservation laws. One merely needs to compute σ_0 and σ_1 in order to find how arbitrary the conservation laws are, with no further prolongation necessary.

When $p = 3$ THEOREM D leads to a very strong result since there is at most one finite relation, namely $\pi = 0$. There are then at least eight variables, $n \geq 8$, so that $n - p \geq 5$, unless some of the variables $\chi_{11}, \chi_{22}, \chi_{33}$ do not appear; in any event $n - p \geq 2$. But, according to the preceding theorem, $\sigma_1 = q$ so that $\sigma_0 = h$. Hence $\sigma_0 + \sigma_1 = n - p \geq 2$, which implies

<u>THEOREM E</u>: For $p = 3$ any system of the form

$$(1.9)\qquad \sum_{\sigma=1}^{3} \left(\alpha^{k1}_\sigma \frac{\partial u^\sigma}{\partial x^1} + \alpha^{k2}_\sigma \frac{\partial u^\sigma}{\partial x^2}\right) = 0, \qquad k = 1,2,3$$

satisfying the conditions of Chapter I, § 3, contains at least a two-parameter family of conservation laws.

§ 6. Several independent variables

The restrictions of Chapter I, § 3, that the original system should contain only two independent variables, that it should be well-determined with distinct characteristic roots, and that its coefficients should be independent of $x = \{x^i\}$, have been very useful, since they permitted the introduction of the normal form of Chapter I, § 4. Without this normal form the method of Chapter II would have been impossible. However, it is possible to use the methods of the present chapter in a way which is completely independent of a normal form so that the preceding restrictions become superfluous. Here we obtain an exterior differential system whose solutions correspond to conservation laws contained in the original system (1.1), with no restrictions whatsoever except that of analyticity, which is required for the Cartan-Kähler theorem. No attempt will be made to decide when the resulting exterior differential system is in involution; in order to do this it would again be wise to introduce some kind of normal form.

System (1.1) is repeated here for convenience as

$$(3.63) \qquad \sum_{i,j=1}^{n,p} \alpha^{ki}_{j} \frac{\partial u^j}{\partial x^i} = 0, \qquad k = 1,\ldots,m.$$

Temporarily we assume that α^{ki}_{j} is an analytic function of $u = \{u^j\}$ alone; this restriction will be removed presently.

Suppose that there exist functions χ_k such that the equation

(3.64) $$\sum_{i,j,k=1}^{n,p,m} \chi_k \alpha_j^{ki} \frac{\partial u^j}{\partial x^i} = 0$$

has the form

(3.65) $$\sum_{i=1}^{n} \frac{\partial \phi^i}{\partial x^i} = 0,$$

that is,

(3.66) $$\sum_{i,j=1}^{n,p} \frac{\partial \phi^i}{\partial u^j} \frac{\partial u^j}{\partial x^i} = 0.$$

Then clearly

(3.67) $$d\Big(\sum_{j=1}^{p} \frac{\partial \phi^i}{\partial u^j} du^j\Big) = dd\phi^i = 0, \quad i = 1,\ldots,n$$

by Poincaré's theorem, so that the necessary and sufficient condition for (3.64) to be a conservation law is that

(3.68) $$d\Big(\sum_{j,k=1}^{p,m} \chi_k \alpha_j^{ki} du^j\Big) = 0, \quad i = 1,\ldots,n,$$

the sufficiency following by the converse of Poincaré's theorem. Thus the existence of conservation laws is reduced to the problem of prolonging (3.68) to a closed system in involution.

Now let us consider an inhomogeneous system

(3.69) $$\sum_{i,j=1}^{n,p} \alpha_j^{ki} \frac{\partial u^j}{\partial x^i} + \alpha^k = 0, \quad k = 1,\ldots,m,$$

where α^{ki}_{j} and α^{k} are analytic functions of $x = \{x^i\}$ as well as u.

If there exist functions χ_k such that

$$(3.70) \qquad \sum_{i,j,k=1}^{n,p,m} \chi_k \alpha^{ki}_{j} \frac{\partial u^j}{\partial x^i} + \sum_{k=1}^{m} \chi_k \alpha^k = 0$$

has the form of a conservation law (3.65), which in this case becomes

$$(3.71) \qquad \sum_{i,j=1}^{n,p} \frac{\partial \phi^i}{\partial u^j} \frac{\partial u^j}{\partial x^i} + \sum_{i=1}^{n} \frac{\partial \phi^i}{\partial x^i} = 0,$$

then clearly

$$(3.72) \qquad d \left(\sum_{j=1}^{p} \frac{\partial \phi^i}{\partial u^j} du^j + \frac{\partial \phi^i}{\partial x^i} dx^i \right) = 0, \qquad i = 1,\ldots,n$$

that is,

$$(3.73) \qquad d \left(\sum_{j,k=1}^{p,m} \chi_k \alpha^{ki}_{j} du^j + \psi_i \, dx^i \right) = 0, \qquad i = 1,\ldots,n$$

is a necessary and sufficient condition for the existence of a conservation law, where

$$(3.74) \qquad \sum_{i=1}^{n} \psi_i = \sum_{k=1}^{m} \chi_k \alpha^k .$$

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